CHAPTER

6

Heat Transfer from Extended Surfaces (FINS)

6.1 Introduction

In this chapter, we shall discuss heat transfer from extended surfaces, also called Fins.

Fins are generally used to enhance the heat transfer from a given surface.

Consider a surface losing heat to the surroundings by convection. Then, the heat transfer rate Q, is given by Newton's Law of Cooling:

$$Q = hA_s(T_s - T_a),$$

where, h = heat transfer coefficient between the surface and the ambient

A_s = exposed area of the surface

T_s = temperature of the surface, and

T_a = temperature of the surroundings.

Now, if we need to increase the heat transfer rate from the surface, we can:

(i) increase the temperature potential, $(T_s - T_a)$; but, this may not be possible always since both these temperatures may not be in our control

(ii) increase the heat transfer coefficient h; this also may not be always possible or it may need installing an external fan or pump to increase the fluid velocity and this may involve cost consideration, or

(iii) increase the surface area A_s; in fact, this is the solution generally adopted. Surface area is increased by adding an 'extended surface' (or, fin) to the 'base surface' by extruding, welding or by simply fixing it mechanically.

Addition of fins can increase the heat transfer from the surface by several folds, e.g. an automobile radiator has thin sheets fixed over the tubes to increase the area several folds and thus increase the rate of heat transfer.

Generally, fins are fixed on that side of the surface where the heat transfer coefficient is low. Heat transfer coefficients are lower for gases as compared to liquids (see Table 1.1). Therefore, one can observe that fins are fixed on the outside the tubes in a car radiator, where cooling liquid flows inside the tubes and air flows on the outside across the fins.

Likewise, in the condenser of a household refrigerator, freon flows inside the tubes and the fins are fixed on the outside of these tubes to enhance the heat transfer rate.

Typical application areas of fins are:

(i) Radiators for automobiles

(ii) Air-cooling of cylinder heads of internal combustion engines (e.g. scooters, motor cycles, aircraft engines), air compressors, etc.

(iii) Economizers of steam power plants

(iv) Heat exchangers of a wide variety, used in different industries

(v) Cooling of electric motors, transformers, etc.

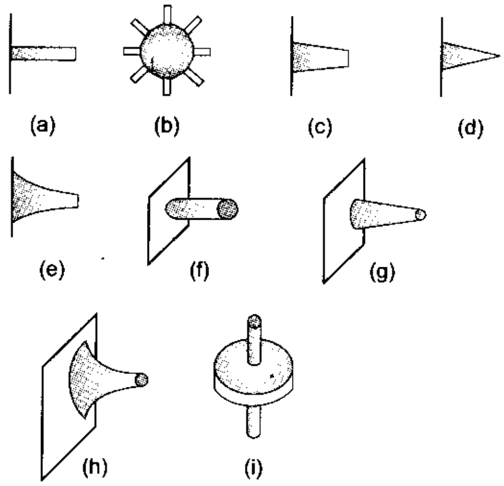


FIGURE 6.1 Different types of fins

(vi) Cooling of electronic equipments, chips, I.C. boards, etc.

(vii) Fin theory is also used to estimate error in temperature measurement while using thermometers or thermocouples.

Types of fins:

There are innumerable types of fins used in practice. Some of the more common types are shown in Fig. 6.1.

A straight fin or spine is an extended surface added to a plane wall. Annular fin is attached circumferentially to a cylinder to increase its surface area. Fins of rectangular, circular, triangular, trapezoidal and conical sections are some of the types commonly used.

Fig. 6.1(a)...longitudinal fin of rectangular profile
Fig. 6.1(b)...cylindrical tube with fins of rectangular profile
Fig. 6.1(c)....longitudinal fin of trapezoidal profile
Fig. 6.1(d)....longitudinal fin of triangular profile
Fig. 6.1(e)....longitudinal fin of parabolic profile
Fig. 6.1(f)....cylindrical pin fin
Fig. 6.1(g)....truncated conical spine
Fig. 6.1(h)....parabolic spine
Fig. 6.1(i)....cylindrical tube with radial fin of rectangular or truncated conical profile.

Cross-sectional areas of annular fins vary with the radius; in contrast, rectangular or cylindrical spines have constant cross-sectional area. Triangular or parabolic fins are used when one optimizes the fins from the view point of weight or volume.

Determination of heat transfer in fins requires information about the temperature profile in the fin. We get the differential equation describing the temperature distribution in the fin by the usual procedure of writing an energy balance for a differential volume of the fin. We shall start by doing this for a fin of uniform cross section.

6.2 Fins of Uniform Cross Section (Rectangular or Circular)—Governing Differential Equation

Let us analyse heat transfer in a fin of rectangular cross section. Same analysis will be valid for a fin of circular cross section also.

Consider a fin of rectangular cross section attached to the base surface, as shown in Fig. 6.2. Let L be the length of fin, w, its width and t, its thickness. Let P be the perimeter = $2(w + t)$. Let A_c be the area of cross section, and, T_0, the temperature at the base, as shown.

Assumptions:

(i) Steady state conduction, with no heat generation in the fin
(ii) Thickness t is small compared to length L and width w, i.e. one-dimensional conduction in the X-direction only.
(iii) Thermal conductivity, k of the fin material is constant.
(iv) Isotropic (i.e. constant k in all directions) and homogeneous (i.e. constant density) material.
(v) Uniform heat transfer coefficient h, over the entire length of fin.
(vi) No bond resistance in the joint between the fin and the base wall, and
(vii) Negligible radiation effect.

Base temperature, T_0 is higher than the ambient temperature, T_a. Temperature will drop along the fin from the base to the tip of the fin, as shown in Fig. 6.2(b). Heat transfer will occur by conduction along the length of the fin and by convection, with a heat transfer coefficient h, from the surface of the fin to the ambient.

Our aim is to derive a differential equation governing the temperature distribution in the fin. Once we get the temperature field, heat flux at any point can easily be obtained by applying Fourier's law.

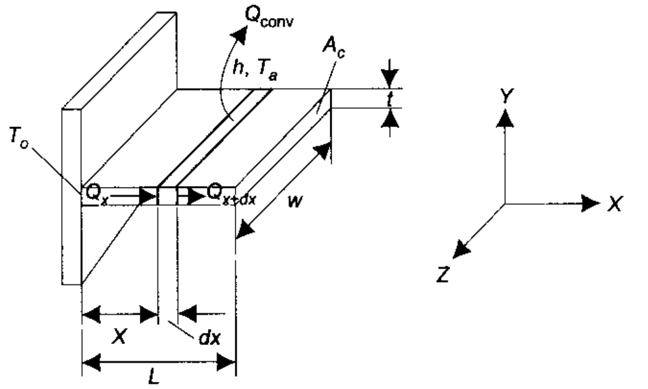


FIGURE 6.2(a) Rectangular fin of uniform cross section

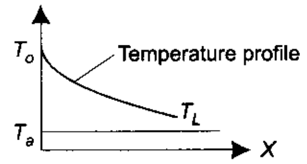


FIGURE 6.2(b) Temperatre profile along length of fin

Consider an elemental section of thickness dx at a distance x from the base as shown. Let us write an energy balance for this element:

Energy going *into* the element by conduction = (Energy *leaving* the element by conduction + Energy *leaving* the surface of the element by convection)

i.e. $$Q_x = Q_{x+dx} + Q_{conv} \quad ...(a)$$

where,

Q_x = heat conducted into the element at x

Q_{x+dx} = heat conducted out of the element at $x + dx$, and

Q_{conv} = heat convected from the surface of the element to ambient

We have:

$$Q_x = -k \cdot A_c \cdot \frac{dt}{dx}$$

$$Q_{x+dx} = -k \cdot A_c \cdot \frac{d}{dx}\left(T + \frac{dT}{dx}.dx\right)$$

i.e. $$Q_{x+dx} = -k \cdot A_c \frac{dT}{dx} - k \cdot A_c \frac{d^2T}{dx^2} \cdot dx$$

and, $$Q_{conv} = h \cdot A_s\,(T - T_a)$$

i.e. $$Q_{conv} = h \cdot (P \cdot dx) \cdot (T - T_a)$$

where, A_s is the surface area of the element P, its perimeter.

Substituting the terms in Eq. a,

$$-k \cdot A_c \cdot \frac{dT}{dx} = \left(-k \cdot A_c \frac{dT}{dx} - k.A_c \frac{d^2T}{dx^2} \cdot dx\right) + h \cdot (P \cdot dx) \cdot (T - T_a)$$

i.e. $$k \cdot A_c \cdot \frac{d^2T}{dx^2} \cdot dx - h \cdot (P \cdot dx) \cdot (T - T_a) = 0$$

i.e. $$\frac{d^2T}{dx^2} - m^2 \cdot (T - T_a) = 0 \quad ...(b)$$

where $$m = \sqrt{\frac{h \cdot P}{k \cdot A_c}}$$

Note that m has units of: (m^{-1}) and is a constant, since for a given operating conditions of a fin, generally h and k are assumed to be constant.

Now, define excess temperature,

$$\theta = T - T_a$$

Therefore, $$\frac{d\theta}{dx} = \frac{dT}{dx} \quad \text{and,} \quad \frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2}$$

since T_a is a constant.

Substituting in Eq. b,

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \qquad ...(6.1)$$

Eq. 6.1 is the governing differential equation for the fin of uniform cross section considered.

Eq. 6.1 is a second order, linear, ordinary differential equation. Its general solution is given by calculus theory, in two equivalent forms:

$$\theta(x) = C_1 \cdot \exp(-m \cdot x) + C_2 \cdot \exp(m \cdot x) \qquad ...(6.2a)$$

where, C_1 and C_2 are constants

and,

$$\theta(x) = A \cdot \cos h(m \cdot x) + B \cdot \sin h(m \cdot x) \qquad ...(6.2b)$$

where A and B are constants, and cos h and sin h are hyperbolic functions, defined in Table 6.1.

Eq. 6.2a or 6.2b describes the temperature distribution in the fin along its length.

To calculate the set of constants C_1 and C_2, or A and B, we need two boundary conditions:

One of the B.C.'s is that the temperature of the fin at its base, i.e. at $x = 0$, is T_o, and this is considered as known.

i.e. B.C. (i): at $x = 0$, $\qquad T = T_o$

Regarding the second boundary condition, there are several possibilities:

Case (i): Infinitely long fin,

Case (ii): Fin insulated at its end (i.e. negligible heat loss from the end of the fin),

Case (iii): Fin losing heat from its end by convection, and

Case (iv): Fin with specified temperature at its end.

It may be remarked here, that while for case (i), it is convenient to choose the solution in the form given by Eq. 6.2a and for cases (ii) and (iii), choosing the solution in the form given by Eq. 6.2b makes the analysis easy.

Before we proceed further, let us tabulate a few useful relations for hyperbolic functions: (See Table 6.1).

6.2.1 *Infinitely Long Fin*

This simply means that the fin is very long. Consequence of this assumption is that temperature at the tip of the fin approaches that of the surrounding ambient as the fin length approaches infinity. See Fig. 6.3 (a).

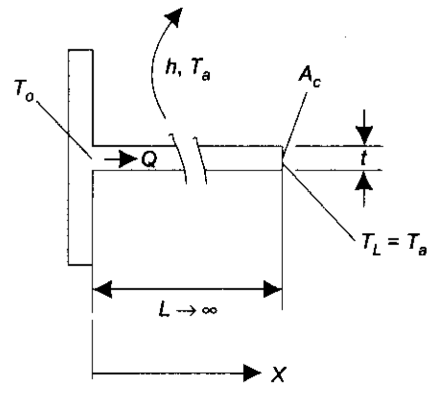


FIGURE 6.3(a) Infinitely long fin of uniform cross section

To determine the temperature distribution:

The governing differential equation, as already derived, is given by Eq. 6.1, i.e.

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \qquad ...(6.1)$$

And, we shall choose for its solution for temperature distribution Eq. (6.2a), i.e.

$$\theta(x) = C_1 \cdot \exp(-m \cdot x) + C_2 \cdot \exp(m \cdot x) \qquad ...(6.2a)$$

C_1 and C_2 are obtained from the B'C.'s:

B.C. (i): $\quad$ at $x = 0$, $T = T_o$

B.C. (ii): as $x \to \infty$, $T \to T_a$, the ambient temperature.

From B.C. (i):

at $x = 0$, $\theta(x) = T_o - T_a = \theta_o$

TABLE 6.1 Relations for hyperbolic functions

Sl. No.	Relation
(a)	$\cos h(\beta) = \dfrac{\exp(\beta) + \exp(-\beta)}{2}$
(b)	$\sin h(\beta) = \dfrac{\exp(\beta) - \exp(-\beta)}{2}$
(c)	$\exp(\beta) + \exp(-\beta) = 2 \cos h(\beta)$
(d)	$\exp(\beta) - \exp(-\beta) = 2 \sin h(\beta)$
(e)	$\exp(\beta) = \cos h(\beta) + \sin h(\beta)$
(f)	$\exp(-\beta) = \cos h(\beta) - \sin h(\beta)$
(g)	$\sin h(0) = 0$
(h)	$\cos h(0) = 1$
(i)	$\dfrac{d}{dx}(\sin h(m \cdot x) = m \cdot \cos h(m \cdot x)$
(j)	$\dfrac{d}{dx} \cos h(m \cdot x) = m \cdot \sin h(m \cdot x)$
(k)	$\cos h(-x) = \cos h(x)$
(l)	$\sin h(-x) = -\sin h(x)$
(m)	$\cos h(x + y) = \cos h(x) \cdot \cos h(y) + \sin h(x) \cdot \sin h(y)$
(n)	$\cos h(x - y) = \cos h(x) \cdot \cos h(y) - \sin h(x) \cdot \sin h(y)$
(o)	$\sin h(x + y) = \sin h(x) \cdot \cos h(y) + \cos h(x) \cdot \sin h(y)$
(p)	$\sin h(x - y) = \sin h(x) \cdot \cos h(y) - \cos h(x) \cdot \sin h(y)$

From B.C. (ii):

at $x = \infty$, $\theta(x) = T_a - T_a = 0$

From B.C. (ii) and Eq. 6.2a: $C_2 = 0$

From B.C. (i) and Eq. 6.2a: $C_1 = \theta_o$

Substituting C_1 and C_2 back in eqn. 6.2a, we get:

$$\theta(x) = \theta_o \cdot \exp(-m \cdot x)$$

i.e. $$\frac{\theta(x)}{\theta_o} = \exp(-m \cdot x)$$

i.e. $$\frac{T(x) - T_a}{T_o - T_a} = \exp(-m \cdot x) \qquad ...(6.3)$$

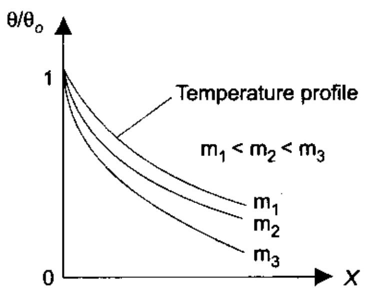


FIGURE 6.3(b) Dimensionless temperature profile along length of fin

Eq. 6.3 gives the temperature distribution in an infinitely long fin of uniform cross section, along the length. This is shown graphically in Fig. 6.3b. Note that temperature distribution is exponential.

It may be observed from the graph that as the parameter m increases, dimensionless temperature ratio falls steeply. As the fin length tends to infinity, dimensionless temperature ratio approaches zero, as shown in the Fig. 6.3(b)

To determine the heat transfer rate:

Heat transfer rate from the fin may be determined by either of the two ways:

(a) by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

i.e. $$Q_{fin} = -k\, A_c dT(x)/dx|_{x=0} = -k\, A_c\, d\theta(x)/dx|_{x=0} \qquad ...(c)$$

(b) by integrating the convective heat transfer for the entire surface of the fin, i.e.

$$Q_{fin} = \int_0^L h \cdot P \cdot (T - T_a)dx = \int_0^L h \cdot P \cdot \theta dx \qquad ...(d)$$

Of course, the results obtained by both the methods must be the same; but applying method (a) is easier.
By method (a):

$$Q_{\text{fin}} = -k\cdot A_c\cdot\left(\frac{d}{dx}T(x)\right)_{x=0} = -k\cdot A_c\cdot\left(\frac{d}{dx}\theta(x)\right)_{x=0} \quad ...(c)$$

i.e.
$$Q_{\text{fin}} = -k\cdot A_c\cdot\left[\frac{d}{dx}\left(\theta_o\cdot e^{-m\cdot x}\right)\right]_{x=0}$$

i.e.
$$Q_{\text{fin}} = -k\cdot A_c\cdot(-m)\cdot\left[\theta_o\cdot\left(e^{-m\cdot x}\right)\right]_{x=0}$$

i.e.
$$Q_{\text{fin}} = k\cdot A_c\cdot m\cdot\theta_o \quad ...(6.4)$$

Substituting for m:

$$Q_{\text{fin}} = k\cdot A_c\cdot\sqrt{\frac{h.P}{k.A_c}}\cdot\theta_o$$

i.e.
$$Q_{\text{fin}} = \sqrt{h\cdot P\cdot k\cdot A_c}\cdot\theta_o = \sqrt{h\cdot P\cdot k\cdot A_c}\cdot(T_o - T_a) \quad ...(6.5)$$

Eq. 6.4 or 6.5 gives the heat transfer rate through the fin.

Let us verify this result from method (b):

By method (b):

$$Q_{\text{fin}} = \int_0^L h\cdot P\cdot(T - T_a)dx = \int_0^L h\cdot P\cdot\theta dx \quad ...(d)$$

i.e.
$$Q_{\text{fin}} = \int_0^\infty h\cdot P\cdot\theta_o\cdot e^{-m.x}\,dx$$

i.e.
$$Q_{\text{fin}} = \frac{1}{m}\cdot h\cdot P\cdot\theta_o$$

i.e.
$$Q_{\text{fin}} = \sqrt{h\cdot P\cdot k\cdot A_c}\cdot\theta_o = \sqrt{h\cdot P\cdot k\cdot A_c}\cdot(T_o - T_a) \quad ...(\textit{same as Eq. 6.5})$$

6.2.2 Fin of Finite Length with Insulated End

End of a fin is generally not insulated; so, here, what we mean is that the heat transfer from the end of the fin is negligible as compared to the heat transfer from the surface of the fin. Mostly, this is true, since the area of the end of fin is negligible compared to the exposed surface area of the fin; in fact, this is the most important case. See Fig. 6.4.

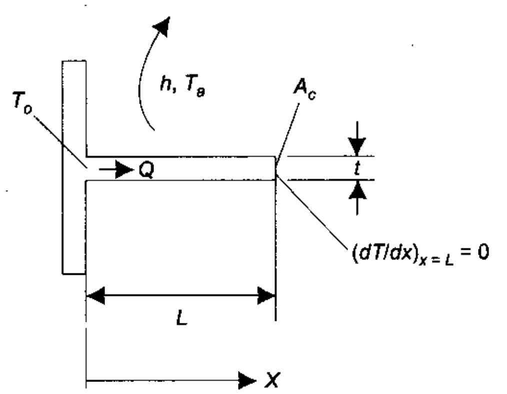


FIGURE 6.4(a) Fin of finite length, end insulated

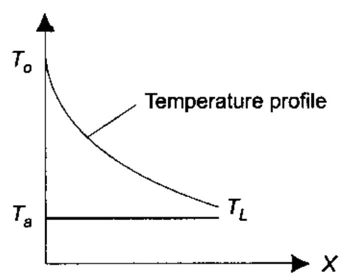


FIGURE 6.4(b) Temperature profile for fin insulated at its end

To determine the temperature distribution:

The governing differential equation., as already derived, is given by Eq. 6.1, namely,

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \qquad ...(6.1)$$

And, we shall choose for its solution for temperature distribution, Eq. 6.2b, i.e.

$$\theta(x) = A \cdot \cos h(m \cdot x) + B \cdot \sin h(m \cdot x) \qquad ...(6.2b)$$

Constants A and B are obtained from the B'C.'s:

B.C.(i):

$$\text{at } x = 0,\ \theta(x) = T_o - T_a = \theta_o$$

B.C. (ii):

$$\text{at } x = L,\ \frac{dT}{dx} = \frac{d\theta}{dx} = 0 \quad \text{since the end is insulated.}$$

From B.C. (i) and Eq. 6.2b:

$$A = \theta_o$$

From B.C. (ii) and Eq. 6.2b:

$$\left(\frac{d\theta}{dx}\right)_{x=L} = 0$$

i.e. $$A \cdot m \cdot \sin h(m \cdot L) + B \cdot m \cdot \cos h(m \cdot L) = 0 \qquad \textit{(using relations in Table 6.1)}$$

Substituting fo A: $\theta_o \cdot (m \cdot \sin h(m \cdot L) + (B \cdot m \cdot \cos h(m \cdot L) = 0)$

i.e. $$B = -\theta_o \cdot \frac{\sin h(m \cdot L)}{\cos h(m \cdot L)}$$

Substituting for A and B in Eq. 6.2b

$$\theta(x) = \theta_o \cdot \cos h(m \cdot x) - \theta_o \cdot \frac{\sin h(m \cdot L)}{\cos h(m \cdot L)} \cdot \sin h(m \cdot x)$$

i.e. $$\frac{\theta(x)}{\theta_o} = \frac{\cos h(m \cdot L) \cdot \cos h(m \cdot x) - \sin h(m \cdot L) \cdot \sin h(m.x)}{\cos h(m \cdot L)}$$

i.e. $$\frac{\theta(x)}{\theta_o} = \frac{\cos h(m \cdot (L - x))}{\cos h(m \cdot L)} \qquad \text{((6.6)...}\textit{using relation no. (n) from Table 6.1}\text{)}$$

i.e. $$\frac{T(x) - T_a}{T_o - T_a} = \frac{\cos h(m \cdot (L - x))}{\cos h(m \cdot L)} \qquad ...(6.7)$$

Eq. 6.6 or 6.7 gives the temperature distribution in the fin with negligible heat transfer from its end.

Same relations are obtained if we start with the general solution for temperature distribution as given by Eq. 6.2a; however, algebraic manipulations required are rather lengthy.

Temperature at the end of the fin:

This is easily determined by putting $x = L$ in Eq. 6.6 or 6.7:

i.e. $$\frac{\theta(L)}{\theta_o} = \frac{1}{\cos h(m \cdot L)} \qquad \text{(6.6a ... }\textit{since cos h}\text{ (0) = 1)}$$

and, $$\frac{T_L - T_a}{T_o - T_a} = \frac{1}{\cos h(m \cdot L)} \qquad ...(6.7a)$$

or, $$T_L = \frac{T_o - T_a}{\cos h(m \cdot L)} + T_a \qquad ...(6.7b)$$

Eq. 6.7b gives the temperature at the end of a fin (i.e. at $x = L$), when the end of the fin is insulated.

To determine the heat transfer rate:

Heat transfer rate from the fin may be determined by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

i.e. $$Q_{fin} = -k\,A_c\,dT(x)/dx\,|_{x=0} = -k\,A_c\,d\theta(x)/dx\,|_{x=0}$$

Therefore,

$$Q_{fin} = -k\cdot A_c\cdot\theta_o\cdot\left[\frac{-m.\sin h(m.(L-x))}{\cos h(m.L)}\right]_{x=0}$$

i.e. $$Q_{fin} = k\cdot A_c\cdot m\cdot\theta_o\cdot\tan h(m\cdot L) \quad ...(6.8)$$

i.e. $$Q_{fin} = \sqrt{h\cdot P\cdot k\cdot A_c}\cdot\theta_o\cdot\tan h(m\cdot L) \quad ...(6.9)$$

Remember: $\theta_o = (T_o - T_a)$.

Eq. 6.8 or 6.9 gives the heat transfer rate from the fin, insulated at its end.

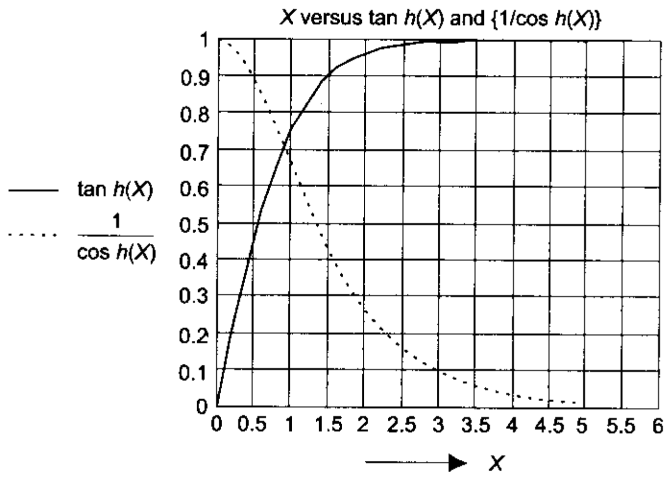
Comparing Eq. 6.8 with that obtained for heat transfer from an infinitely long fin, i.e. Eq. 6.4, we see that a fin with insulated end becomes equivalent to an infinitely long fin when $\tan h(m.L) = 1$.

Table 6.2 below shows values of $\tan h(m.L)$ for values of $(m.L)$ ranging from 0 to 5; same table is also shown in graphical form on the right, for easy visualisation.

It is observed from the Table 6.2 that when $(m\cdot L)$ for the insulated-end fin reaches a value of about 2.8, heat transfer rate becomes about 99% of that obtained for an infinitely long fin. And, beyond a value of $(m\cdot L)$ more

TABLE 6.2 Values of tan h(X) for different values of X

X	tan h(X)
0	0
0.2	0.19738
0.4	0.37995
0.6	0.53705
0.8	0.66404
1	0.76159
1.2	0.83365
1.4	0.88535
1.6	0.92167
1.8	0.94681
2	0.96403
2.2	0.97574
2.4	0.98367
2.6	0.98903
2.8	0.99263
3	0.99505
3.2	0.99668
3.4	0.99777
3.6	0.99851
3.8	0.999
4	0.99933
4.2	0.99955
4.4	0.9997
4.6	0.9998
4.8	0.99986
5	0.99991



Note: $X = m\cdot L$

$$m = \sqrt{\frac{h\cdot p}{k\cdot A_c}}$$

than 5, the fin with insulated end can be considered as infinitely long. Therefore, from the heat transfer point of view, there is no great advantage in having a fin with $(m \cdot L)$ greater than 2.8 or 3.

6.2.3 Fin of Finite Length Losing Heat from its End by Convection

This is a more realistic case, though the relations developed are a little more complicated, as we shall see presently. See Fig. 6.5.

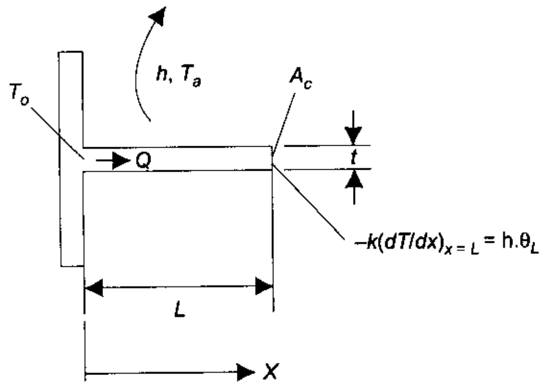


FIGURE 6.5(a) Fin of finite length, end losing heat by convection

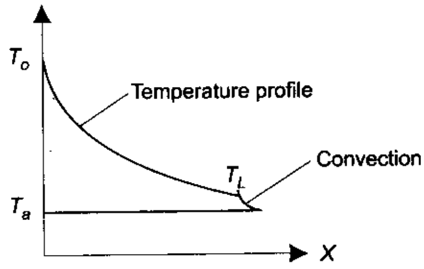


FIGURE 6.5(b) Temperature profile for fin losing heat at its end

Here, heat conducted to the tip of the fin must be equal to the heat convected away from the tip to the ambient, i.e.

$$-k \cdot A_c \cdot \left(\frac{dT}{dx}\right)_{x=L} = h \cdot A_c \cdot (T_L - T_a)$$

i.e.
$$-k \cdot \left(\frac{dT}{dx}\right)_{x=L} = h \cdot \theta_L$$

To determine the temperature distribution:

The governing differential equation., as already derived, is given by Eq. 6.1, i.e.

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \qquad ...(6.1)$$

And, we shall choose for its solution for temperature distribution, Eq. 6.2b, i.e.

$$\theta(x) = A \cdot \cos h(m \cdot x) + B \cdot \sin h(m \cdot x) \qquad ...(6.2b)$$

Constants A and B are obtained from the B'C.'s:

B.C.(i):

$$\text{at } x = 0, \quad \theta(x) = T_o - T_a = \theta_o$$

Applying B.C.(i) to Eq. 6.2b:

$$A = \theta_o$$

B.C. (ii): at $x = L$,

heat conducted to the end = heat convected from the end

i.e.
$$-k \cdot A_c \cdot \left(\frac{d\theta(x)}{dx}\right)_{x=L} = h \cdot A_c \cdot \theta(L) \text{ where } \theta(L) = T_L - T_a$$

i.e.
$$k \cdot \left(\frac{d\theta(x)}{dx}\right)_{x=L} + h \cdot \theta(L) = 0$$

i.e. $$k\cdot[A\cdot m\cdot \sin h(m\cdot L) + B\cdot m\cdot \cos h(m\cdot L)] + h\cdot[A\cdot \cos h(m\cdot L) + B\cdot \sin h(m.L)] = 0$$

(using relations in Table 6.1)

Substituting for A:

$$A\cdot\{(m\cdot k\cdot \sin h(m\cdot L) + h\cdot \cos h(m\cdot L)\} + B\cdot\{(m\cdot k\cdot \cos h(m\cdot L) + h.\sin h(m\cdot L)\}$$

i.e. $$B = \frac{-\theta_o\cdot\left(\sin h(m\cdot L) + \frac{h}{m\cdot k}\cdot\cos h(m\cdot L)\right)}{\cos h(m\cdot L) + \frac{h}{m\cdot k}\cdot\sin h(m\cdot L)}, \text{ since } A = \theta_o.$$

Now, substitute for A and B in the general solution given by Eq. 6.2:

$$\frac{\theta(x)}{\theta_o} = \cos h(m\cdot x) - \frac{\left(\sin h(m\cdot L) + \frac{h}{m\cdot k}\cdot\cos h(m\cdot L)\right)}{\cos h(m\cdot L) + \frac{h}{m.k}\cdot\sin h(m\cdot L)}\cdot\sin h(m\cdot x)$$

i.e. $$\frac{\theta(x)}{\theta_o} = \frac{\left(\cos h(m\cdot x)\cdot\cos h(m\cdot L) + \frac{h}{m.k}\cdot\cos h(m\cdot x)\cdot\sin h(m\cdot L)\right) - \sin h(m\cdot x)\cdot\sin h(m\cdot L) - \frac{h}{m.k}\cdot\cos h(m\cdot L)\cdot\sin h(m\cdot x)}{\cos h(m\cdot L) + \frac{h}{m\cdot k}\cdot\sin h(m\cdot L)}$$

i.e. $$\frac{\theta(x)}{\theta_o} = \frac{[\cos h(m\cdot L).\cos h(m\cdot x) - \sin h(m\cdot L)\cdot\sin h(m\cdot x)] + \frac{h}{m\cdot k}\cdot[\sin h(m\cdot L)\cdot\cos h(m\cdot x) - \cos h(m\cdot L).\sin h(m\cdot x)]}{\cos h(m\cdot L) + \frac{h}{m\cdot k}\cdot\sin h(m\cdot L)}$$

i.e. $$\frac{\theta(x)}{\theta_o} = \frac{\cos h(m\cdot(L-x)) + \frac{h}{m\cdot k}\cdot\sin h(m\cdot(L-x))}{\cos h(m\cdot L) + \frac{h}{m\cdot k}\cdot\sin h(m\cdot L)}$$ ((6.10)...*using relations (n) and (p) from Table* 6.1)

Eq. 6.10 gives the temperature distribution in a fin losing heat by convection at its end.
Remember again that:

$$\theta(x) = T(x) - T_a$$

and, $$\theta_o = T_o - T_a$$

Note that when $h = 0$, i.e. for negligible heat transfer at the tip of the fin, Eq. 6.10 reduces to Eq. 6.6, for a fin with insulated tip.

To determine the heat transfer rate:

Heat transfer rate from the fin may be determined by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

i.e. $$Q_{fin} = -k\,A_c\,[dT(x)/dx]_{x=0} = -k\,A_c\,d\theta(x)/dx]_{x=0}$$

i.e. $$Q_{fin} = -k\cdot A_c\cdot\theta_o\cdot\frac{\left(-m\cdot\sin h(m\cdot L) - \frac{h}{m\cdot k}\cdot m\cdot\cos h(m\cdot L)\right)}{\left(\cos h(m.L) + \frac{h}{m\cdot k}\cdot\sin h(m\cdot L)\right)}$$

i.e. $$Q_{fin} = k\cdot A_c\cdot m\cdot\theta_o\cdot\frac{\left(\sin h(m\cdot L) + \frac{h}{m\cdot k}\cdot\cos h(m\cdot L)\right)}{\left(\cos h(m\cdot L) + \frac{h}{m\cdot k}\cdot\sin h(m\cdot L)\right)}$$

i.e. $$Q_{\text{fin}} = k \cdot A_c \cdot m \cdot \theta_0 \cdot \frac{\left(\tan h(m.L) + \frac{h}{m.k}\right)}{\left(1 + \frac{h}{m.k} \cdot \tan h(m \cdot L)\right)} \qquad ...(6.11)$$

Eq. 6.11 gives the heat transfer rate from a fin losing heat by convection at its tip.

Note: Eq. 6.11 is important since it represents the heat transfer rate for a practically important case of a fin losing heat from its end. However, it is rather complicated to use. So, in practice, even when the fin is losing heat from its tip, it is easier to use Eq. 6.8 or 6.9 obtained for a fin with insulated tip, but with a corrected length, L_c rather than the actual length, L, to include the effect of convection at the tip. In that case, *only to evaluate Q*, L is replaced by a corrected length L_c, in Eq. 6.8 or 6.9, as follows:

For rectangular fins: $L_c = L + \frac{t}{2}$ where t is the thickness of fin

For cylindrical (round) fins: $L_c = L + \frac{r}{2}$ where r is the radius of the cylindrical fin.

6.2.4 Fin of Finite Length with Specified Temperature at its End

This type of problem occurs very often in practice, e.g. when a structural member is used as a heat shunt between two heat reservoirs. Then, the problem is to find out the heat transfer through that member.

Let us formulate the problem as follows:

Problem. A thin fin of length L has its two ends attached to two parallel walls, maintained at temperatures T_1 and T_2, as shown in Fig. 6.6. The fin loses heat by convection to the ambient air at T_a. Assuming one-dimensional conduction, derive an expression for temperature distribution in the fin. Then, deduce an expression for the heat lost by the fin.

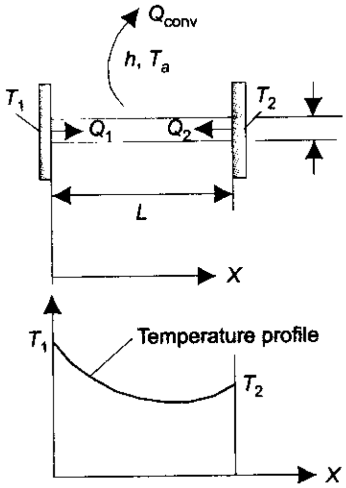


FIGURE 6.6 Fin of finite length, with specified temperature at two ends and the temperature profile along the length

To determine the temperature distribution:

The governing differential equation., as already derived, is given by Eq. 6.1, i.e.

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \qquad ...(6.1)$$

And, we shall choose for its solution for temperature distribution, Eq. 6.2b i.e.

$$\theta(x) = A \cdot \cos h(m \cdot x) + B \cdot \sin h(m \cdot x) \qquad ...(6.2b)$$

Constants A and B are obtained from the B'C.'s:

B.C.(i): at $x = 0$: $T = T_1$ i.e. $\theta = \theta_1$

B.C.(ii): at $x = 0$: $T = T_2$ i.e. $\theta = \theta_2$

From B.C.(i) and Eq. 6.2b:

$$A = \theta_1$$

From B.C.(ii) and Eq. 6.2b:

$$\theta_2 = A \cdot \cos h(m \cdot L) + B \cdot \sin h(m \cdot L)$$

i.e. $$\theta_2 = \theta_1 \cdot \cos h(m \cdot L) + B \cdot \sin h(m \cdot L)$$

Therefore, $$B = \frac{\theta_2 - \theta_1 \cdot \cos h(m \cdot L)}{\sin h(m \cdot L)}$$

Substituting for A and B in Eq. 6.2b:

$$\theta(x) = \theta_1 \cdot \cos h(m \cdot x) + \frac{\theta_2 - \theta_1 \cdot \cos h(m \cdot L)}{\sin h(m \cdot L)} \cdot \sin h(m \cdot x)$$

i.e. $$\theta(x) = \frac{\theta_1 \cdot \sin h(m \cdot L) \cdot \cos h(m \cdot x) - \theta_1 \cdot \cos h(m \cdot L) \cdot \sin h(m \cdot x) + \theta_2 \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)}$$

i.e. $$\theta(x) = \frac{\theta_1 \cdot \sin h(m\cdot(L-x)) + \theta_2 \cdot \sin h(m\cdot x)}{\sin h(m\cdot L)}$$ ((6.12)...*using relation (p) from Table* 6.1)

Eq. 6.12 gives the temperature distribution along the length of fin, when its two ends are maintained at two specified temperatures.

To determine the heat transfer rate:

Total heat transfer rate from the fin is determined by integrating the convection heat transfer over the length of the fin:

$$Q_{fin} = \int_0^L h\cdot(P\cdot dx)\cdot(T(x) - T_a) = \int_0^L h\cdot(P\cdot dx)\cdot\theta(x)$$

i.e. $$Q_{fin} = \int_0^L h\cdot P\cdot\theta(x)dx$$

i.e. $$Q_{fin} = h\cdot P\cdot\int_0^L \frac{\theta_1 \sin h(m\cdot(L-x)) + \theta_2 \cdot \sin h(m\cdot x)}{\sin h(m\cdot L)}dx$$

i.e. $$Q_{fin} = \frac{hP}{\sin h(mL)}\left[\frac{-\theta_1 \cos h(m(L-x))}{m} + \frac{\theta_2 \cos h(mx)}{m}\right]_0^L$$

i.e. $$Q_{fin} = \frac{h\cdot P}{\sin h(mL)}\cdot\left[\frac{-\theta_1}{m}\cdot(1 - \cos h(m\cdot L)) + \frac{\theta_2}{m}\cdot(\cos h(m\cdot L) - 1)\right]$$

i.e. $$Q_{fin} = \frac{h\cdot P}{m\cdot\sin h(m\cdot L)}\cdot[(\theta_1 + \theta_2)\cdot(\cos h(m\cdot L) - 1)]$$

But, $$m = \sqrt{\frac{h\cdot P}{k\cdot A_c}}$$

Therefore, substituting for m:

$$Q_{fin} = \sqrt{h\cdot P\cdot k\cdot A_c}\cdot(\theta_1 + \theta_2)\cdot\left(\frac{\cos h(m\cdot L) - 1}{\sin h(m\cdot L)}\right)$$

i.e. $$Q_{fin} = k\cdot A_c\cdot m\cdot(\theta_1 + \theta_2)\cdot\left(\frac{\cos h(m\cdot L) - 1}{\sin h(m\cdot L)}\right)$$ ...(6.13)

Eq. 6.13 gives the heat transfer rate for a fin with specified temperatures at its both ends.

To find the minimum temperature in the fin:

Differentiate the expression for $\theta(x)$, i.e. Eq. 6.12 w.r.t. x and equate to zero; solving it, we get x_{min}, the position where minimum temperature occurs. Then, substitute this value of x_{min} back in Eq. 6.12 to get the value of T_{min}. (Remember: $\theta(x) = T(x) - T_a$).

When both the ends of fin are at the same temperature:

Now, $T_1 = T_2$ (i.e. $\theta_1 = \theta_2$), and obviously, the minimum temperature will occur at the centre, i.e. at $x = L/2$.

Then, substituting $\theta_1 = \theta_2$ and $x = L/2$ in Eq. 6.12, we get for minimum temperature:

$$\theta(x) = \frac{\theta_1 \cdot \sin h(m\cdot(L-x)) + \theta_2 \cdot \sin h(m\cdot x)}{\sin h(m\cdot L)}$$ ...(6.12)

Therefore, $$\theta_{min} = \frac{\theta_1 \cdot \sin h\left[m\cdot\left(L - \frac{L}{2}\right)\right] + \theta_1 \cdot \sin h\left(m\cdot\frac{L}{2}\right)}{\sin h(m\cdot L)}$$

i.e. $$\theta_{min} = \frac{2\cdot\theta_1 \cdot \sin h\left(\frac{m\cdot L}{2}\right)}{\sin h(m\cdot L)}$$ ...(6.14)

Remember: $\theta_{min} = T_{min} - T_a$

Let us make two important notes:

Note 1: Pin fin (or, spine) of uniform cross section:

See Fig. 6.7. All the above analysis for a fin of rectangular cross section shown in Fig. 6.2, is valid for a fin of uniform circular cross section too.

Note 2: Fin parameter, *m*:

It may be observed from all the derivations for fins done so far, that the parameter m occurs in all the equations. By definition,

$$m = \sqrt{\frac{h \cdot P}{k \cdot A_c}}$$

where, h is the heat transfer coefficient between the fin surface and the ambient, A_c is the cross-sectional area of the fin, P is the perimeter of the fin section and, k is the thermal conductivity of the fin material. Units of m is: m^{-1}.

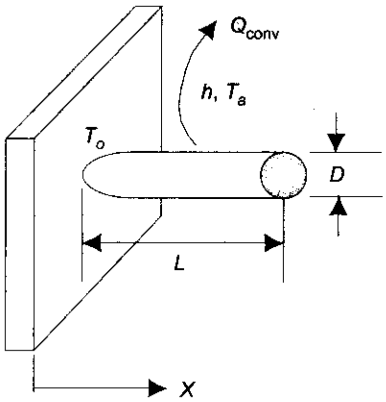


FIGURE 6.7 Fin of circular (round) cross section

(a) For rectangular fin of Fig. 6.2:

We get:

$$A_c = w \cdot t$$

$$P = 2 \cdot (w + t)$$

where, w = width of fin and t = thickness of fin

Therefore,

$$m = \sqrt{\frac{h \cdot P}{k \cdot A_c}}$$

i.e.

$$m = \sqrt{\frac{2 \cdot h \cdot (w + t)}{k \cdot w \cdot t}}$$

Then, for thin fins, i.e. $w << t$, we can write:

$$m = \sqrt{\frac{2 \cdot h}{k \cdot t}} \quad \textit{(for thin fins)}$$

(b) For round fin (or pin fin) of Fig. 6.7:

In this case,

$$A_c = \frac{\pi \cdot D^2}{4}$$

$$P = \pi \cdot D$$

where, D is the diameter of the fin.

Therefore,

$$m = \sqrt{\frac{h \cdot P}{k \cdot A_c}} = \sqrt{\frac{h \cdot \pi \cdot D}{k \cdot \frac{\pi \cdot D^2}{4}}}$$

i.e.

$$m = \sqrt{\frac{4 \cdot h}{k \cdot D}} \quad \textit{(for round (or pin) fins)}$$

6.2.5 Summary of Fin Formulae

The foregoing results, derived for fins with different boundary conditions at the tip, are summarized in Table 6.3 for easy reference.

Example 6.1. (a) A very long, 25 mm diameter copper rod (k = 380 W/(mC)), extends horizontally from a plane heated wall at 150°C. Temperature of surrounding air is 30°C and heat transfer coefficient between the surface of the rod and the surroundings is 10 W/(m^2K).

(i) Determine the rate of heat loss from the rod

(ii) How long the rod should be to be considered as infinite?

(iii) Draw the temperature profile along the length of the rod.

TABLE 6.3 Temperature distribution and heat transfer rate for fins of uniform cross section
*(x) = (T(x) − Ta), m = /{h·P/(k·Ac)}

Case	Tip condition ($x = L$)	Temperature distribution, $\theta(x)/\theta_o$	Heat transfer rate, Q_{fin}
1	Infinitely long $L \to \infty$, $\theta(L) = 0$	$\frac{\theta(x)}{\theta_o} = \exp(-m\cdot x)$	$Q_{fin} = k\cdot A_c\cdot m\cdot \theta_o$
2	Insulated at the tip $(d\theta/dx)\vert_{x=L} = 0$	$\frac{\theta(x)}{\theta_o} = \frac{\cos h(m\cdot(L-x))}{\cos h(m\cdot L)}$	$Q_{fin} = k\cdot A_c\cdot m\cdot \theta_o\cdot \tan h(m\cdot L)$
3	Convection from tip $-k\frac{d\theta}{dx}\Big\vert_{x=L} = h\cdot\theta(L)$	$\frac{\theta(x)}{\theta_o} =$ $\frac{\cos h(m\cdot(L-x)) + \frac{h}{m\cdot k}\cdot \sin h(m\cdot(L-x))}{\cos h(m\cdot L) + \frac{h}{m\cdot k}.\sin h(m\cdot L)}$	$Q_{fin} = k\cdot A_c\cdot m\cdot \theta_o x$ $\frac{\left(\tan h(m\cdot L) + \frac{h}{m\cdot k}\right)}{\left(1 + \frac{h}{m\cdot k}\cdot \tan h(m\cdot L)\right)}$
4(a)	Prescribed temperatures at the tip, ends $x = 0 \to \theta = \theta_1$ $x = L \to \theta = \theta_2$	$\theta(x) = \frac{\theta_1 \sin h((m\cdot(L-x)) + \theta_2\cdot \sin h(m\cdot x)}{\sin h(m\cdot L)}$	$Q_{fin} = k\cdot A_c\cdot m\cdot(\theta_1 + \theta_2)x$ $\left(\frac{\cos h(m\cdot L) - 1}{\sin h(m\cdot L)}\right)$
4(b)	When temperatures at both ends are equal, $T_1 = T_2$ or, $\theta_1 = \theta_2$	$\theta(x) = \frac{\theta_1 \sin h(m\cdot(L-x)) + \theta_1 \sin h(m\cdot x)}{\sin h(m\cdot L)}$ Minimum temperature is given by: $\theta_{min} = \frac{2\cdot\theta_1\cdot \sin h\left(\frac{m\cdot L}{2}\right)}{\sin h(m\cdot L)}$	$Q_{fin} = k\cdot A_c\cdot m\cdot(2\cdot\theta_1).$ $\left(\frac{\cos h(m\cdot L) - 1}{\sin h(m\cdot L)}\right)$

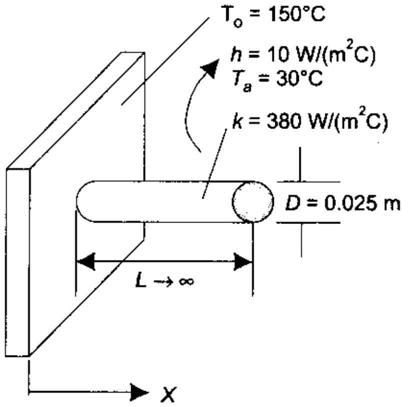


FIGURE Example 6.1 Fin of circular (round) cross section

(b) Compare the temperature distribution in the rod if the materials were: (i) copper (k = 380 W/(mC)), (ii) aluminium (k = 200 W/(mC)) and, (iii) steel (k = 55 W/(mC)). Other data is the same as in part (a).

Solution. Since it is stated that it is a very long rod, we will take L as ∞. So, relations derived for an infinitely long fin apply.

See Fig. Example 6.1.

Data:

$D := 0.025$ m $L := \infty$ m $k := 380$ W/(mC) $T_o := 150$°C

$T_a := 30$°C $h := 10$ W/(m^{2}C)

Heat transfer rate from the rod:

First, let us calculate the parameter m:

We have: $m = \sqrt{\frac{h\cdot P}{k\cdot A_c}}$ where, P is the perimeter and A_c is the area of cross section.

Then, $A_c := \frac{\pi\cdot D^2}{4}$ m^2 *(define the area of cross section of the rod)* i.e. $A_c = 4.909 \times 10^{-4}$ m^2 *(area of cross section of the rod)*

and, $P := \pi \cdot D$, m *(define the perimeter of the rod)*

i.e. $P = 0.079$ m *(perimeter of the rod)*

Therefore, $m := \sqrt{\frac{h \cdot P}{k \cdot A_c}}$ m^{-1} *(define the parameter m.)*

i.e. $m = 2.052$ m^{-1} *(parameterm.)*

Now, apply Eq. 6.4 for heat transfer from a a very long fin:

$$Q_{fin} = k \cdot A_c \cdot m \cdot \theta_o \qquad ...(6.4)$$

i.e. $Q_{fin} := k \cdot A_c \cdot m \cdot (T_o - T_a)$

Substituting values: $Q_{fin} = 45.931$ W ***(heat loss rate from the fin.)***

Length of rod required to consider it as infinitely long:

Read the discussion under section 6.2.2.

An infinitely long fin has no heat transfer from its end since the end temperature tends to the ambient temperature as the length tends to infinity. Therefore, comparing the expressions for Q for an infinitely long fin and a fin with insulated at its end, i.e.

$$Q_{fin} = k \cdot A_c \cdot m \cdot \theta_o \qquad \textit{(for infinitely long fin)}$$

$$Q_{fin} = k \cdot A_c \cdot m \cdot \theta_o \cdot \tan h(m \cdot L) \qquad \textit{(for a fin with insulated end,)}$$

we see that they are equivalent when $\tan h(m \cdot L)$ is equal to 1.

From Table 6.2, it is seen that at $(m \cdot L) = 5$, $\tan h(m \cdot L)$ is almost equal to 1.

Therefore, the rod can be considered as infinitely long, if: $m \cdot L > 5$, or $L > (5/m)$:

i.e. $L := \frac{5}{m}$ m *(define L)*

i.e. $L = 2.437$ m ***(length of rod rquired to consider it as infinitely long.)***

To draw the temperature profile in the rod:

We need the equation for temperature profile.

Eq. 6.3 gives the temperature profile for a very long fin:

i.e.
$$\frac{T(x) - T_a}{T_o - T_a} = \exp(-m \cdot x) \qquad ...(6.3)$$

Therefore, $T(x) := T_a + (T_o - T_a) \cdot \exp(-m \cdot x)$ *(equation for temperature profile in the rod)*

We use Mathcad to draw the temperature profile. First, define a range variable x, varying from 0 to say, 2.5 m, with an increment of 0.1 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. (Fig. Ex. 6.1, b)

$x := 0, 0.1, ..., 2.5$ *(define a range variable x.. starting value = 0, next value = 0.1 m, and last value = 2.5 m)*

Observe from the graph that at $x = 0$, the temperature is 150°C and at the end of the fin, the temperature is 30°C which is that of the ambient. This matches with the boundary conditions of the problem.

(b) Compare the temperature distribution if the fin materials are aluminium (k = 200 W/(mC)) and stainless steel (k = 55 W/(mC)):

For aluminium: $k_{A1} := 200$ W/(m·C) *(thermal conductivity of aluminium)*

Therefore, $m_{A1} := \sqrt{\frac{h \cdot P}{k_{A1} \cdot A_c}}$ m^{-1} *(define fin parameter m for aluminium)*

i.e. $m_{A1} = 2.828$ m^{-1}

and corresponding length required to be considered as infinitely long is:

$L := \frac{5}{m_{A1}}$ i.e. $L = 1.768$ m

For steel: $k_s := 55$ W/(m·C) *(thermal conductivity of steel)*

Therefore, $m_s := \sqrt{\frac{h \cdot P}{k_s \cdot A_c}}$ m^{-1} *(define fin parameter m for steel.)*

i.e. $m_s = 5.394$ m^{-1}

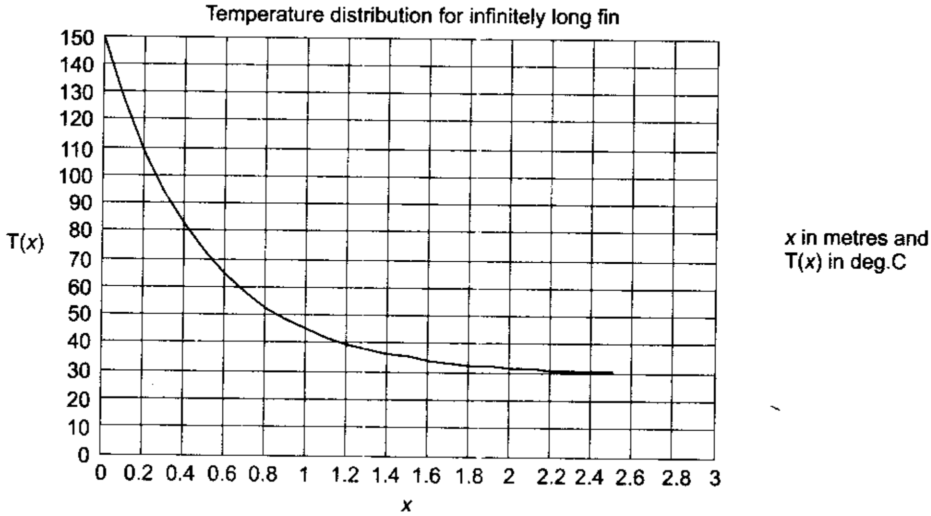


FIGURE Example 6.1(b)

and, corressponding length required to be considered as infinitely long is:

$$L := \frac{5}{m_s} \quad \text{i.e.} \quad L = 0.927 \text{ m}$$

To draw the temperature profiles for the three materials:

First, define T as a function of x and m:

i.e. $T(x, m) := T_a + (T_o - T_a)\cdot \exp(m \cdot x)$ *(equation for temperature profile in the rod)*

Then, we have: $m_{cu} := 2.052$ *...parameter m for copper...calculated in part* (a)

$m_{Al} = 2.828$ *...parameter m for aluminium*

$m_s = 5.394$ *...parameter m for steel*

We use Mathcad to draw the temperature profiles. First, define a range variable x, varying from 0 to say, 3.0 m, with an increment of 0.1 m. Then, choose x–y graph from the graph palette, and fill up the place holder on the x-axis with x; and in the place holder on the y-axis, fill up $T(x, m_{cu})$, $T(x, m_{Al})$, $T(x, m_s)$ separated by commas as shown. Click anywhere outside the graph region, and immediately the graphs appear. (Fig. Ex. 6.1, c)

$x := 0, 0.1, \ldots, 3$ *(define a range variable x.. starting value = 0, next value = 0.1 m, and last value = 3.0 m)*

It may be observed from the above graph that:

(i) higher the thermal conductivity, higher is the steady state temperature attained at a given location.

(ii) to attain the same temperature on the rod, longer length is required for a material of higher thermal conductivity.

(iii) it is verified that fin can be considered as infinitely long if the lengths are 2.437, 1.768 and 0.927 m for copper, aluminium and steel, respectively, i.e. the end temperature becomes equal to the ambient temperature at these lengths.

Example 6.2. To determine the thermal conductivity of a long, solid 2.5 cm diameter rod, one half of the rod was inserted to a furnace while the other half was projecting into air at 27°C. After steady state had been reached, the temperatures at two points 7.6 cm apart were measured and found to be 126°C and 91°C, respectively. The heat transfer coefficient over the surface of the rod exposed to air was estimated to be 22.7 W/(m^{2}K). What is the thermal conductivity of the rod?

Solution. See Fig. Example 6.2.

Data:

$D := 0.025$ m $\quad h := 22.7$ W/(m^2 K) $\quad T_a := 27$°C

Since it is a very long (or, infinitely long) rod, we can take any point as the origin. But, taking point A as origin as shown in the Fig. 6.9 simplifies the solution. Then, we write:

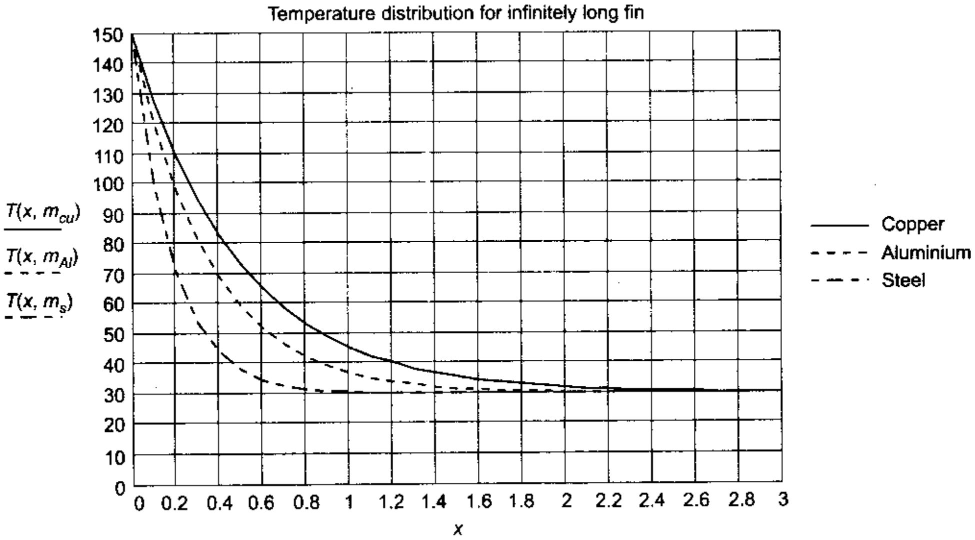


FIGURE Example 6.1(c)

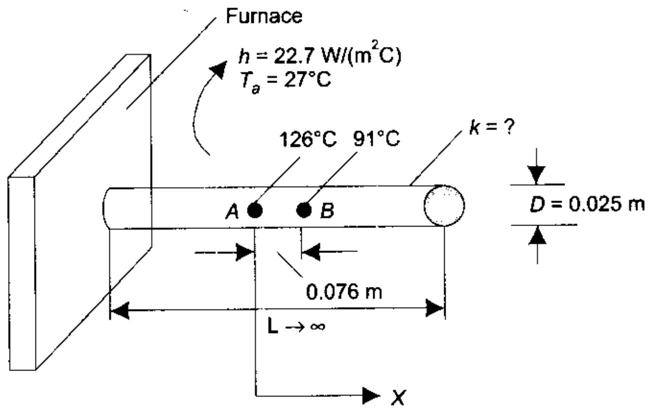


FIGURE Example 6.2 Very long fin of circular cross section

$x := 0.076$ m (*distance of point B from origin (i.e. point A)*)

$T_o := 126°C$ (*temperature at the origin (point A)*)

$T(x) := 91°C$ (*temperature at point B.*)

Temperature distribution in a long fin is given by Eq. 6.3:

i.e. $$\frac{T(x) - T_a}{T_o - T_a} = \exp(-m \cdot x) \qquad ...(6.3)$$

Therefore, $$m := \frac{-\ln\left(\frac{T(x) - T_a}{T_o - T_a}\right)}{x} \text{ m}^{-1}$$ (*define fin parameter m*)

i.e. $m = 5.74 \text{ m}^{-1}$

But, we have, by definition of m:

$$m = \sqrt{\frac{h \cdot P}{k \cdot A_c}}$$ (*Eq. (A) definition of fin parameter m*)

In the present case,

	$P := \pi \cdot D$, m	*(define P, the perimeter)*
i.e.	$P = 0.079$ m	*(perimenter)*
and,	$A_c := \frac{\pi \cdot D^2}{4}$, m^2	*(define area of cross section A_c)*
i.e.	$A_c = 4.909 \times 10^{-4}$ m^2	*(area of cross section A_c)*

Therefore, from Eq. (A): $k: = \frac{h \cdot P}{A_c \cdot m^2}$ W/(mK) define thermal conductivity

i.e. $k = 110.237$ W/(mK) ...***value of thermal conductivity.***

Example 6.3. Aluminum square fins (0.5 mm × 0.5 mm) of 1 cm length are provided on the surface of an electronic semiconductor device to carry 46 mW of energy generated by the electronic device and the temperature at the surface of the device should not exceed 80°C. The temperature of the surrounding medium is 40°C. Thermal conductivity of aluminium =190 W/(mK) and heat transfer coefficient h = 12.5 W/(m^{2}K). Find number of fins required to carry out the above duty. Neglect the heat loss from the end of the fin. [M.U.]

Solution. This is the case of fin, insulated at its end, since by data, there is no heat loss from the end of the fin. Therefore, Eq. 6.7 for temperature distribution and Eq. 6.8 for heat transfer rate, are applicable.

See Fig. Example 6.3.

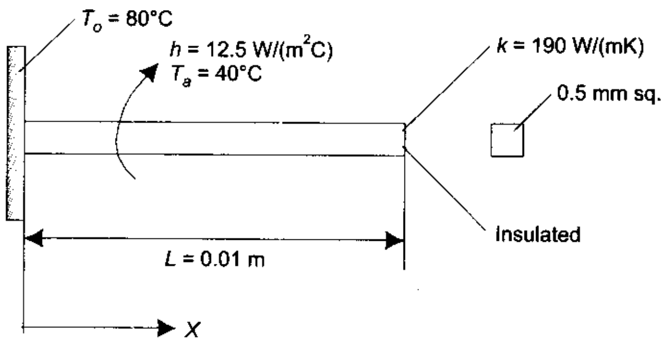


Figure Example 6.3 Finite fin insulated at its tip

Date:

Q_{tot} := 0.046 W L: = 0.01 m w := 0.0005 m t := 0.0005 m k := 190 W/(m K) T_o := 80°C

T_a := 40°C h := 12.5 W/(m^2 K)

Let us first calculate heat transferred from one fin; then, knowing the total amount of heat to be transferred, we can find out the total number of fins required.

Fin parameter m:

We have	$m = \sqrt{\frac{h \cdot P}{k \cdot A_c}}$	*(fin parameter)*
Now,	$A_c := w \cdot t$, m^2	*(define area of cross section of fin)*
i.e.	$A_c = 2 \cdot 5 \times 10^{-7}$ m^2	*(area of cross section of fin)*
and,	$P := 2 \cdot (w + t)$, m	*(define perimeter of fin sectior)*
i.e.	$P = 2 \times 10^{-3}$ m	*(perimeter of fin section)*
Therefore,	$m := \sqrt{\frac{h \cdot P}{k \cdot A_c}}$ m^{-1}	*(define fin parameter m)*
i.e.	$m = 22.942$ m^{-1}	*(fin parameter m)*
Also,	$\theta_o := T_o - T_a$ °C	*(define excess temperature at the base)*
i.e.	$\theta_o = 40$°C	*(θ at the base, i.e. at $x = 0$)*

Applying Eq. 6.8 for heat transfer rate from a fin with insulated end:

i.e. $Q_{fin} := k \cdot A_c \cdot m \cdot \theta_o \cdot \tanh(m \cdot L)$ ...(6.8)

we get: $Q_{fin} = 9.82818 \times 10^{-3}$ W *(heat transfer per fin)*

Therefore, number of fins required to carry 46 mW:

$$N = \frac{Q_{tot}}{Q_{fin}} = 4.68$$ ***(i.e. say, 5 fins.)***

Example 6.4. A cylinder 5 cm diameter and 50 cm long, is provided with 14 longitudinal straight fins of 1 mm thick and 2.5 mm height. Calculate the heat loss from the cylinder per second if the surface temperature of the cylinder is 200°C. Take h = 25W/(m^2 K), k = 80 W/(mK), and T_a = 45°C. [M.U.]

Solution. See Fig. Example 6.4.

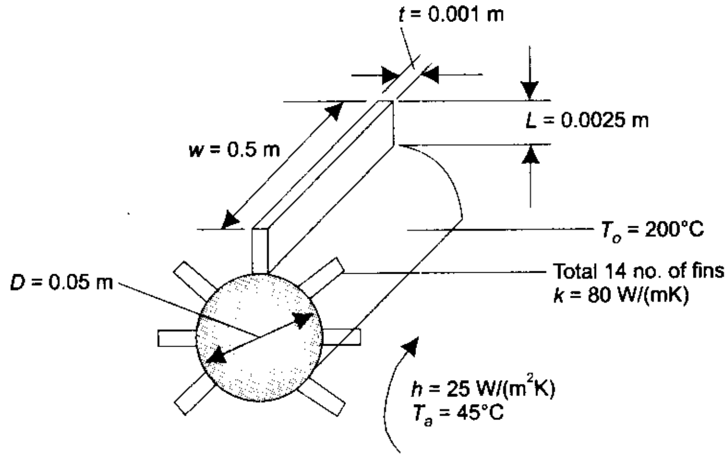


FIGURE Example 6.4 Longitudinal fins on a cylinder, losing heat from tip

This is the case of a fin with convection from its end. Therefore, Eq. 6.10 for temperature distribution and Eq. 6.11 for heat transfer rate, are applicable. However, Eq. 6.11 is a little complicated to use; so, as remarked earlier, we shall use the Eq. 6.8 for a fin with insulated end, but with the modification that the corrected length, L_c is used instead of L. Then, we will check the result thus obtained, by applying Eq. 6.11.

Total heat transfer is calculated as the sum of heat transferred from all the 14 fins and the convective heat transfer from the unfinned base surface of the cylinder, which is at a temperature of 200°C.

Data:

$L := 0.0025$ m $w := 0.5$ m $t := 0.001$ m $N := 14$ $D := 0.05$ m $k := 80$ W/(m K) $T_o := 200$°C

$T_a := 45$°C $h := 25$ W/(m^{2}K) $\theta_o := T_o - T_a$ °C i.e. $\theta_o = 155$°C

Fin parameter *m*:

We have: $m = \sqrt{\frac{h \cdot P}{k \cdot A_c}}$

Now, $A_c := w \cdot t$, m^2 *(define area of cross section of fin)*

i.e. $A_c = 5 \times 10^{-4}$ m^2 *(area of cross section of fin)*

and, $P := 2 \cdot (w + t)$, m *(define perimeter of fin section)*

i.e. $P = 1.002$ m *(perimeter of fin section)*

Therefore, $m := \sqrt{\frac{h \cdot P}{k \cdot A_c}}$ m^{-1}, *(define fin parameter m)*

i.e. $m = 25.025$ m^{-1} *(fin parameter m)*

Corrected length, L_c:

For rectangular cross section:

$$L_c := L + \frac{t}{2}, \text{ m} \qquad \textit{(define corrrected length)}$$

i.e. $L_c = 3 \times 10^{-3}$ m *(corrected length.)*

Heat transfer from fins:

Applying Eq. 6.8 for heat transfer rate from a fin with insulated end, using L_c instead of L:

i.e. $Q_{fin} := k \cdot A_c \cdot m \cdot \theta_o \cdot \tan h(m \cdot L_c)$ ...(6.8)

we get: $Q_{fin} = 11.62642$ W *(heat transfer per fin)*

Therefore,

heat transfer from 14 fins: $Q_{tot1} := Q_{fin} \cdot 14$ W *(define heat transfer from 14 fins)*

i.e. $Q_{tot1} = 162.77$ W *(heat transfer from **14** fins.)*

Heat transfer from unfinned surface of cylinder:

Unfinned surface area: $A_{unfin} := (\pi \cdot D - N \cdot t) \cdot w$, m^2 = subtracting the area occupied by 14 fins from the surface area of the cylinder

i.e. $A_{unfin} = 0.072$ m^2 *...unfinned area*

Therefore, $Q_{tot2} := h \cdot A_{unfin} \cdot (T_o - T_a)$ W *...heat transfer from unfinned base area*

i.e. $Q_{tot2} = 277.217$ W *...heat transfer from unfinned base area*

Total heat transfer rate:

$Q_{total} := Q_{tot1} + Q_{tot2}$ W *...define total heat transfer*

i.e. $Q_{total} = 439.987$ W *....**total heat transfer.***

Verify: Let us check the result obtained by using Eq. 6.8 by comparing it with the result that would be obtained if we use the accurate relation for heat transfer for fin with convection from its end, i.e. Eq. 6.11:

$$Q_{fin} := k \cdot A_c \cdot m \cdot \theta_o \cdot \frac{\left(\tan h(m \cdot L) + \frac{h}{m \cdot k}\right)}{\left(1 + \frac{h}{m \cdot k} \cdot \tan h(m \cdot L)\right)} \qquad \text{...(6.11)}$$

i.e. $Q_{fin} = 11.62266$ W *(heat transfer per fin)*

This value compares very well with the result obtained from Eq. 6.8, i.e. 11.626 W.

Example 6.5. Two ends of a copper rod (k = 380 W/(mK)), 15 mm diameter and 300 mm long are connected to two walls, each maintained at 300°C. Air is blown across the rod with a heat transfer coefficient of 20 W/(m^{2}K). Air temperature is 40°C. Determine:

(i) mid-point temperature of the rod
(ii) net heat transfer to air
(iii) heat transferred from the first 0.1 m of the rod from LHS. [M.U.]

Solution. See Fig. Example 6.5.

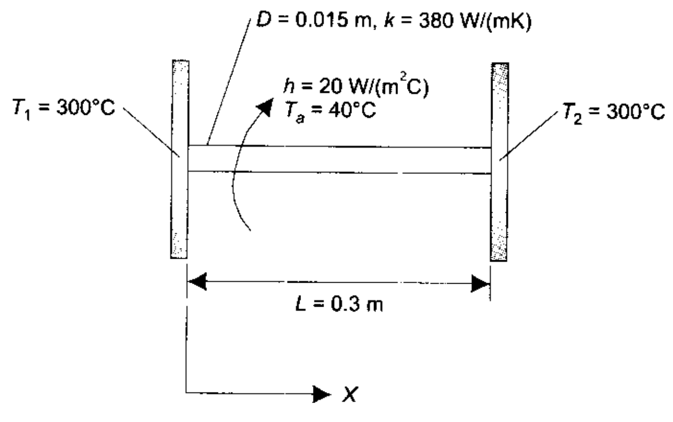


FIGURE Example 6.5 Fin with equal temperature at both ends

This is the case of a fin with specified temperatures at its both ends. So, we can directly use Eq. 6.12 for temperature distribution and Eq. 6.13 for heat transfer rate.

However, **easier method** to solve, is as follows: since the temperatures at both ends are same, it is immediately clear that the minimum temperature occurs at the mid-point. i.e. at the mid-point, dT/dx is equal to zero; but, this is also the condition for an insulated end. Therefore, the given rod of length L may be considered as made up of two fins, each of length $L/2$, insulated at its end. So, for one half of the rod, we can apply the simpler Eq. 6.7 for temperature distribution and Eq. 6.8 for heat transfer, for a fin with insulated end.

We will verify the result later, by using Eq.6.12 and 6.13.

Data:

$D := 0.015$ m $L := 0.3$ m $k := 380$ W/(m K) $T_1 := 300°C$ $T_2 := 300°C$

$T_a := 40°C$ $h := 20W/(m^2K)$

Fin parameter m:

We have: $$m = \sqrt{\frac{h \cdot P}{k \cdot A_c}}$$

Now, $A_c := \frac{\pi \cdot D^2}{4}$, m^2 *(define area of cross section of fin)*

i.e. $A_c = 1.767 \times 10^{-4}$ m^2 *(area of cross section of fin)*

and, $P := \pi \cdot D$, m *(define perimeter of fin section)*

i.e. $P = 0.047$ m *(perimeter of fin section)*

Therefore, $m := \sqrt{\frac{h \cdot P}{k \cdot A_c}}$ m^{-1}, *(define fin parameter m)*

i.e. $m = 3.746$ m^{-1}, *(fin parameter m.)*

Mid-point temperature of rod:

Now, left half of the rod can be considered as a fin of length $L/2$, with its end insulated.

So, for temperature distribution, apply Eq. 6.7, putting $L = L/2$, $T_o = T_1$

$$\frac{T(x) - T_a}{T_o - T_a} = \frac{\cos h(m \cdot (L - x))}{\cos h(m \cdot L)} \quad ...(6.7)$$

Putting $L = L/2$:

$$T(x) := (T_1 - T_a) \cdot \frac{\cos h\left[m \cdot \left(\frac{L}{2} - x\right)\right]}{\cos h\left(m \cdot \frac{L}{2}\right)} + T_a$$ *(Eq. A...temperature distribution in the rod)*

Therefore, mid-point temperature is obtained by putting $x = L/2$:

i.e. $T(0.15) = 263.734°C$ ***(mid-point temperature.)***

To draw the temperature profile:

We use Mathcad to draw the temperature profile. First, define a range variable x, varying from 0 to 0.3 m, with an increment of 0.01 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 6.5(b)

$x := 0, 0.01, ..., 0.3$ *(define a range variable x..starting value = 0, next value = 0.01 m, and last value = 0.3 m)*

It may be verified from the graph that temperature at both the ends is 300°C and the minimum. temperature occurs at mid-point (i.e. $x = 0.15$ m), with $T_{min} = 263.73°C$.

Also, note that temperature distribution as given in Eq. A plots the temperature distribution over the whole length since beyond $x = L/2 = 0.15$ m, in the numerator of first term in Eq. A, the relation $\cos h(-x) = \cos h(x)$ applies, and beyond the mid-point, we get a mirror image of the graph on the left.

Heat transfer:

Heat transfer for the first half of the rod is given by Eq. 6.8. Total heat transfer from the rod is, of course, **twice** this value:

i.e. $$Q_{fin} = k \cdot A_c \cdot m \cdot \theta_o \cdot \tan h(m \cdot L) \quad ...(6.8)$$

Note that in Eq. 6.8, we have to put $L = L/2$, $\theta_o = (T_1 - T_a)$

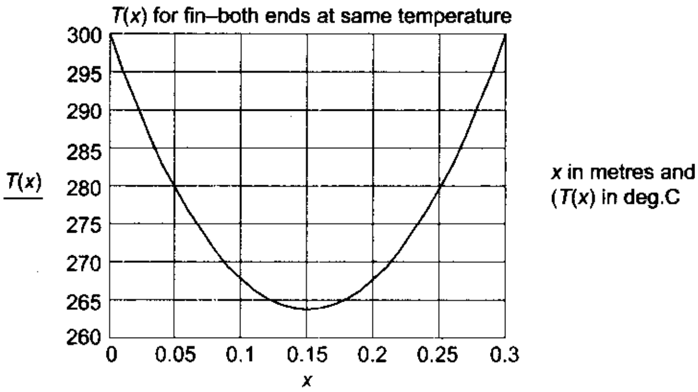


FIGURE Example 6.5(b)

Therefore, total heat transfer from the rod:

$$Q_{\text{total}} := 2 \cdot k \cdot A_c \cdot m \cdot (T_1 - T_a) \cdot \tan h\left(m \cdot \frac{L}{2}\right) \quad ...(B)$$

i.e. $Q_{\text{total}} = 66.642$ W *(**heat transfer from the rod**)*

Verify: Let us verify the results now from Eqs. 6.12 and 6.13.

For temperature profile: use Eq. 6.12:

i.e.
$$\theta(x) = \frac{\theta_1 \cdot \sin h(m \cdot (L - x)) + \theta_2 \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)} \quad ...(6.12)$$

Put here: $\theta_1 = \theta_2 = T_1 - T_a$

$$\theta(x) := \frac{(T_1 - T_a) \cdot \sin h(m \cdot (L - x)) + (T_1 - T_a) \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)} \quad ...(C)$$

and, at mid-point, $x = \frac{L}{2} = 0.15$ m

Therefore, $\theta(0.15) = 223.734$°C *(excess temperature = T(x) – T(a))*

i.e. $T(0.15) = 223.734 + 40 = 263.734$°C *(temperature at mid-point...**same as obtained earlier.**)*

or, directly from Eq. 6.14:

i.e.
$$\theta_{\min} = \frac{2 \cdot \theta_1 \cdot \sin h\left(\frac{m \cdot L}{2}\right)}{\sin h(m \cdot L)} \quad ...(6.14)$$

i.e.
$$T_{\min} := T_a + \frac{2 \cdot (T_1 - T_a) \cdot \sin h\left(\frac{m \cdot L}{2}\right)}{\sin h(m \cdot L)}$$

i.e. $T_{\min} = 263.734$ C *(temperature at mid-point ...**same as obtained above.**)*

For heat transfer: use Eq. 6.13:

$$Q_{\text{fin}} = k \cdot A_c \cdot m \cdot (\theta_1 + \theta_2) \cdot \left(\frac{\cos h(m \cdot L) - 1}{\sin h(m \cdot L)}\right) \quad ...(6.13)$$

Put here: $\theta_1 = \theta_2 = T_1 - T_a = 260$

i.e.
$$Q_{\text{fin}} := k \cdot A_c \cdot m \cdot (260 + 260) \cdot \left(\frac{\cos h(m \cdot L) - 1}{\sin h(m \cdot L)}\right) \quad ...(D)$$

i.e. $Q_{fin} = 66.642$ W *(heat transfer from rod...**same as obtained earlier.**)*

Heat transfer from the first 0.1 m from LHS:

To get this, integrate the convective heat transfer from $x = 0$ to $x = 0.1$ using for $T(x)$, Eq. A.

$$T(x) := (T_1 - T_a) \cdot \frac{\cos h\left[m \cdot \left(\frac{L}{2} - x\right)\right]}{\cos h\left(m \cdot \frac{L}{2}\right)} + T_a$$ (eq. (A)...*temperature distribution in the rod*)

$$Q := h \cdot \pi \cdot D \cdot \int_0^{0.1} (T(x) - T_a)\,dx$$ (eq. (E)...*define Q, heat transfer from a length of 0.1 m from LHS*)

i.e. $Q = 22.716$ W ***(heat transfer from a length of 0.1 m from LHS.)***

Note: In Eq. E, we have used Newton's Law of Cooling, i.e. $Q = h\ A\ \Delta T$. Elemental area involved was $P.dx = \pi.D.dx$ and $\Delta T = (T_1 - T_a)$. Also, note that while using Mathcad, the calculation of integral within the prescribed limits is returned directly; there is no need to do the labour of expanding the integral and substituting the limits.

Example 6.6. (a) In Example 6.6, if the two ends of the rod are maintained at 300°C and 260°C, respectively, determine:

(i) location and value of minimum temperature in the rod
(ii) mid-point temperature of the rod
(iii) draw the temperature profile
(iv) net heat transfer to air
(v) heat transferred from the first 0.1 m length of the rod from LHS
(vi) heat transferred from the left end (i.e. at $x = 0$)

(b) If in this example, if there is an uniform heat generation $q_g = 1.5 \times 10^5$ W/m^3 in the rod, determine:

(i) location and value of minimum temperature in the rod
(ii) mid-point temperature of the rod
(iii) draw the temperature profile.

Solution. See Fig. Example 6.6.

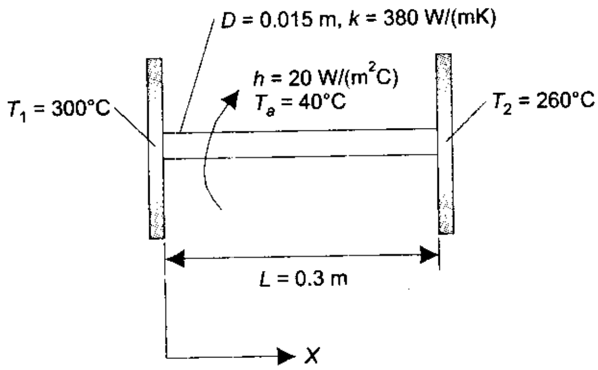


FIGURE Example 6.6a Fin wth different temperture at the two ends

Data:

$T_2 := 260°C$ *...temperature on RHS*

Rest of the data is same as given in Example 6.6.

This is the case of a fin with specified temperatures at its both ends. So, we can use Eq. 6.12 for temperature distribution and Eq. 6.13 for heat transfer rate.

However, let us work out this problem from fundamentals, and then verify the result from Eqs. 6.12 and 6.13.

Fin parameter *m*:

We have: $$m = \sqrt{\frac{h \cdot P}{k \cdot A_c}}$$

This is already worked out to be: $m = 3.746\ m^{-1}$ *(fin parameter m)*

Now, as shown in section 6.2.1, the controlling differential equation for this problem is:

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \quad ...(6.1)$$

And, for its solution, let us choose Eq. 6.2b:

$$\theta(x) = A \cdot \cos h(m \cdot x) + B \cdot \sin h(m \cdot x) \quad ...(6.2b)$$

Eq. 6.2b gives the temperature profile in the fin. Constants A and B are determined from the boundary conditions:

B.C. (i): at $x = 0$, $\theta(0) = \theta_1$

B.C. (ii): at $x = L$, $\theta(L) = \theta_2$

Then, from B.C. (i) and Eq. 6.2b:

$$A = \theta_1 = T_1 - T_a$$

i.e. $A = 260$

and, from B.C.(ii) and Eq. 6.2b:

$$\theta_2 = (T_2 - T_a) = A \cdot \cos h(m \cdot L) + B \cdot \sin h(m \cdot L)$$

i.e. $$B := \frac{(T_2 - T_a) - A \cdot \cos h(m \cdot L)}{\sin h(m \cdot L)}$$ *(define constant B)*

i.e. $B = -161.52177$ *(value of constant B)*

Therefore, equation for temperature distribution is:

$$\theta(x) := 260 \cdot \cos h(m \cdot x) - 161.5218 \sin h(m \cdot x) \quad ...(A)$$

Location and value of minimum temperature:

Differentiate Eq. A w.r.t. x and equate to zero; solving, we get the location x_{min} of minimum temperature from LHS. Then, substitute $x = x_{min}$ in Eq. A to get the value of minimum temperature

Let $d\theta(x)/dx$ be defined as $\theta'(x)$.

$$\theta'(x) = \frac{d\theta(x)}{dx} = 260 \cdot m \cdot \sin h(m \cdot x) - 161.5218\ m \cdot \cos h(m \cdot x) = 0$$

Solving, $$\tan h(m \cdot x) = \frac{161.5218}{260} = 0.621$$

i.e. $m \cdot x = a \tan h(0.621) = 0.727$ *(a tan h means inverse of tan h)*

i.e. $$x_{min} = \frac{0.727}{m} = \frac{0.727}{3.746} = 0.194 \text{ m}$$ ***(location of minimum temperature in the rod.)***

Now, substitute this value of x_{min} in Eq. A to get value of T_{min}:

i.e. $\theta(0.194) = 203.742°C$ *...value of $\theta(min)$*

i.e. $T_{min} := 203.742 + T_a$ *...since $\theta(x) = T(x) - T_a$*

i.e. $T_{min} = 243.742°C$ ***(minimum temperature in the rod.)***

Temperature at mid-point:

Put $x = 0.15$ m in Eq. A:

$\theta(0.15) = 206.524$ *(excess temperature at mid-point)*

Therefore, $T_{mid} := 206.524 + T_a$ °C *(since $\theta(x) = T(x) - T_a$)*

i.e. $T_{mid} = 246.524°C$ ***(temperature at mid-point of rod.)***

To draw the temperature profile:

We use Mathcad to draw the temperature profile. First, define a range variable x, varying from 0 to 0.3 m, with an increment of 0.01 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. below.

We have:

$$\theta(x) := 260 \cdot \cos h(m \cdot x) - 161.5218 \sin h(m \cdot x) \quad ...(A)$$

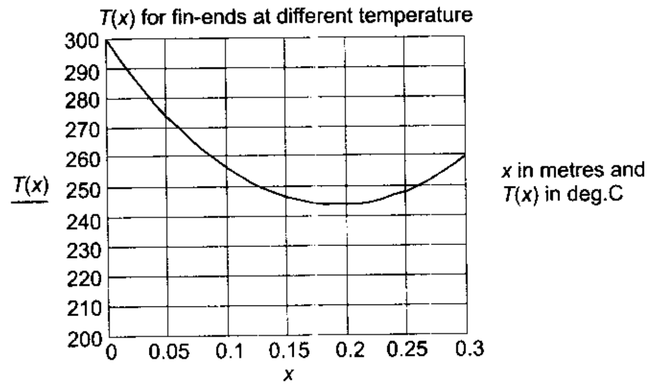
Therefore, $T(x) = \theta(x) + T_a$ *(define $T(x)$, temperature at any point in the rod)*

$x := 0, 0.01, ..., 0.3$ *(define a range variable x...starting value = 0, next value = 0.01 m, and last value = 0.3 m)*

Note from the graph that the end temperatures are 300°C and 260°C, and temperature at mid-point is 246.52°C; also, the minimum temperature occurs at $x = 0.194$ m, and its value is 243.74°C.

Net heat transfer from the rod:

This is obtained by integrating the convective heat transfer over the entire surface of the rod:



$$Q := \int_0^{0.3} h \cdot P \cdot \theta(x) dx \ \text{W}$$ *(from Newton's law, $Q = hA\Delta T$; $A = P.dx$, and $\theta(x) = T(x) - T_a$)*

i.e. $Q = 61.515$ W ***...heat transfer from the rod.***

Heat transfer from the first 0.1 m of length of rod:

This is obtained by integrating the convective heat transfer from $x = 0$ to $x = 0.1$ m:

$$Q := \int_0^{0.1} h \cdot P \cdot \theta(x) dx \ \text{W}$$ *(from Newton's law, $Q = hA\Delta T$; $A = P.dx$, and $\theta(x) = T(x) - T_a$)*

i.e. $Q = 22.197$ W ***(heat transfer from first 0.1 m of the rod.)***

Now, verify from Eqs. 6.12 and 6.13:

We have, for temperature distribution:

$$\theta(x) := \frac{\theta_1 \cdot \sin h(m \cdot (L - x)) + \theta_2 \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)} \quad ...(6.12)$$

$\theta_1 := T_1 - T_a$ i.e. $\theta_1 = 260$

$\theta_2 := T_2 - T_a$ i.e. $\theta_2 = 220$

Therefore, $\theta(0.15) = 206.524$°C *(excess temperature at mid-point)*

i.e. $T_{mid} := 206.524 + T_a$

i.e. $T_{mid} = 246.524$°C ***(temperature at mid-point...verified.)***

And, for heat transfer:

$$Q_{fin} := k \cdot A_c \cdot m \cdot (\theta_1 + \theta_2) \cdot \left(\frac{\cos h(m \cdot L) - 1}{\sin h(m \cdot L)} \right) \quad ...(6.13)$$

i.e. $Q_{fin} = 61.515$ W ***(heat transfer from the rod...verfied.)***

Heat transferred from the left end (i.e. at $x = 0$):

Q_{left} is calculated by applying the Fourier's law at $x = 0$.

We already have the equation for temperature distribution, i.e.

$$\theta(x) = 260 \cdot \cos h(m \cdot x) - 161.5218 \sin h(m \cdot x) \quad ...(A)$$

$$Q_{left} = -k \cdot A_c \cdot \left(\frac{d\theta(x)}{dx} \right)_{x=0}$$ *(applying Fourier's law at the left end)*

i.e. $Q_{left} := -k \cdot A_c \cdot (0 - 161.5218 \ m \cdot 1)$ W *(since $\sin h(0) = 0$ and $\cos h(0) = 1$)*

i.e. $Q_{left} = 40.634$ W ***(heat transferred from left end.)***

Check: Let us check this result also by calculating heat transferred from the right end; sum of heat transferred from left and right ends must be equal to 65.515 W, calculated earlier.

Let $\theta'(x) := \frac{d}{dx}\theta(x)$ *(define the first derivative of $\theta(x)$ w.r.t. x)*

$Q_{right} := -k \cdot A_c \cdot \theta'(0.3)$ *(applying Fourier's law at $x = L$, i.e. $x = 0.3$ m)*

i.e. $Q_{right} = -20.881$ W ***(heat transferred from the right end.)***

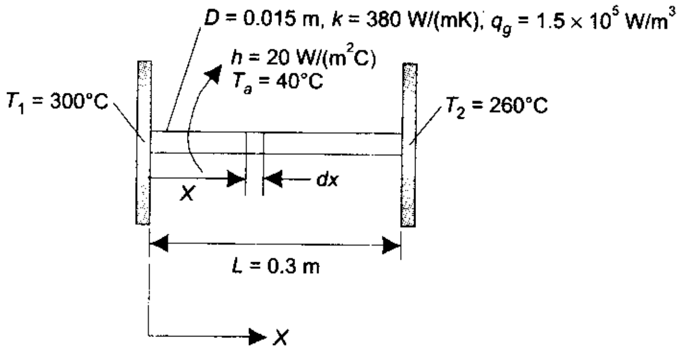


FIGURE Example 6.6(b) Fin with heat generation, two ends at prescribed temperature

Remark: Negative sign indicates that heat transfer is from right to left, i.e. in negative X-direction. Also, note that now, we have performed the differentiation and put $x = 0.3$ m, directly in Mathcad, instead of doing it by long hand.

Adding, $|Q_{left}| + |Q_{right}| = 61.515$ W *(total heat transferred...checks with earlier result.)*

(b) If there is uniform heat generation, q_g (W/m³) in the rod:

See Fig. Example 6.6(b).

Let us derive the governing differential equation by the usual method of making an energy balance on a differential element of the rod of length dx at a distance x from the origin, as shown in the Fig. 6.6(b).

We write:

Energy into the element from left face + heat generated in the element =

Energy out at the right face + Energy lost by convection from the surface of the element

i.e.

$$-k \cdot A_c \cdot \frac{dT}{dx} + q_g \cdot A_c \cdot dx = \left(-k \cdot A_c \frac{dT}{dx} - k \cdot A_c \cdot \frac{d^2T}{dx^2} \cdot dx \right) + h \cdot (P \cdot dx) \cdot (T - T_a)$$

i.e.

$$-k \cdot A_c \cdot \frac{d^2T}{dx^2} \cdot dx + h \cdot (P \cdot dx) \cdot (T - T_a) - q_g \cdot A_c \cdot dx = 0$$

i.e.

$$k \cdot A_c \cdot \frac{d^2T}{dx^2} - h \cdot P \cdot (T - T_a) + q_g \cdot A_c = 0$$

i.e.

$$\frac{d^2T}{dx^2} - m^2 \cdot (T - T_a) + \frac{q_g}{k} = 0 \qquad \text{...(a)}$$

where, $m = \sqrt{\dfrac{h \cdot P}{k \cdot A_c}}$

Substituting $\theta = T - T_a$ we get:

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta + \frac{q_g}{k} = 0 \qquad \text{...(b)}$$

Now, make another subsitution: $\theta' = \theta - \dfrac{q_g}{k \cdot m^2}$

Then, Eq. b becomes:

$$\frac{d^2\theta'}{dx^2} - m^2 \cdot \theta' = 0 \qquad \text{...(c)}$$

General solution of Eq. c is:

$$\theta'(x) = A \cdot \cos h(m \cdot x) + B \cdot \sin h(m \cdot x) \qquad \text{...(d)}$$

A and B constants, determined from Boundary Conditions, i.e.

B.C. (i): at $x = 0, \ \theta'(0) = \theta'_1$

B.C. (i) and Eq. d gives: $\theta'_1 = A$

B.C. (ii): at $x = L, \ \theta'(L) = \theta'_2$

B.C. (ii) and Eq. d gives:

$$\theta'_2 = \theta'_1 \cdot \cos h(m \cdot L) + B \cdot \sin h(m \cdot L)$$

i.e. $$B = \frac{\theta'_2 - \theta'_1 \cdot \cos h(m \cdot L)}{\sin h(m \cdot L)}$$

Substituting for A and B in Eq. d:

$$\theta'(x) = \theta'_1 \cos h \cdot (m \cdot x) + \frac{\theta'_2 - \theta'_1 \cdot \cos h(m \cdot L)}{\sin h(m \cdot L)} \cdot \sin h(m \cdot x)$$

i.e. $$\theta'(x) = \frac{\theta'_1 \cdot (\cos h(m \cdot x) \cdot \sin h(m \cdot L) - \sin h(m \cdot x) \cdot \cos h(m \cdot L)) + \theta'_2 \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)}$$

i.e. $$\theta'(x) = \frac{\theta'_1 \cdot (\sin h(m \cdot (L - x))) + \theta'_2 \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)}$$ *(using eqn. (p) from Table 6.1...(e))*

Remembering that: $\theta' = \theta - \frac{q_g}{k \cdot m^2}$ and, $\theta = T - T_a$

we get: $$T(x) = T_a + \frac{q_g}{k \cdot m^2} + \frac{\theta'_1 \cdot (\sin h(m \cdot (L - x))) + \theta'_2 \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)} \quad ...(f)$$

Eq. f gives the desired temperature profile in the rod when there is uniform heat generation in the rod and the ends are maintained at prescribed temperatures.

Now, for the present case:

$q_g = 1.5 \times 10^5 \ W/m^3$ *(heat generation rate)*

$\theta'_1 = T_1 - T_a - \frac{q_g}{k \cdot m^2}$ i.e. $\theta'_1 = 231.875$

$\theta'_2 = T_2 - T_a - \frac{q_g}{k \cdot m^2}$ i.e. $\theta'_2 = 191.875$

Therefore, from Eq. f:

$$\text{Temp}(x) := T_a + \frac{q_g}{k \cdot m^2} + \frac{\theta'_1 \cdot (\sin h(m \cdot (L - x))) + \theta'_2 \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)}$$ ...(f)...*define Temp (x)*

Minimum temperature in the rod:

Differentiate Eq. f w.r.t. x and equate to zero: solving, get the location of minimum temperature, x_{min}. Substitute this value of x back in Eq. f to get T_{min}:

Let $\text{Temp}'(x) := \frac{d}{dx}\text{Temp}(x)$ *(define Temp'(x) as the first derivative of Temp(x) w.r.t. x)*

Use solve block to solve Temp'(x) = 0. Start with a trial value of x, write the constraint immediately below 'Given'; then the typing Find (x) immediately gives the value of x_{min}.

$x := 0.1$ m *(trial value of x)*

Given

Temp'(x) = 0

Find(x) = 0.2

i.e. $x_{min} := 0.2$ m ***(location of minimum temperature in the rod.)***

Substitute this x_{min} back in Eq. f to get T_{min}:

Temp(x_{min}) = 247.289°C ***(minimum temperature in the rod, with heat generation.)***

Temperature at mid-point:

At mid-point, $x := 0.15$:

Temp(0.15) = 250.447°C ***(temperature at mid-point, when there is heat generation in the rod.)***

To draw temperature profile in the rod:

We use Mathcad to draw the temperature profile. First, define a range variable x, varying from 0 to 0.3 m, with an increment of 0.01 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis with x and on y-axis, with $T(x)$, Temp(x), respectively. Click any where outside the graph region, and immediately the graph appears. It contains two curves, one for x vs. $T(x)$ and the other x vs. Temp(x). See fig. below.

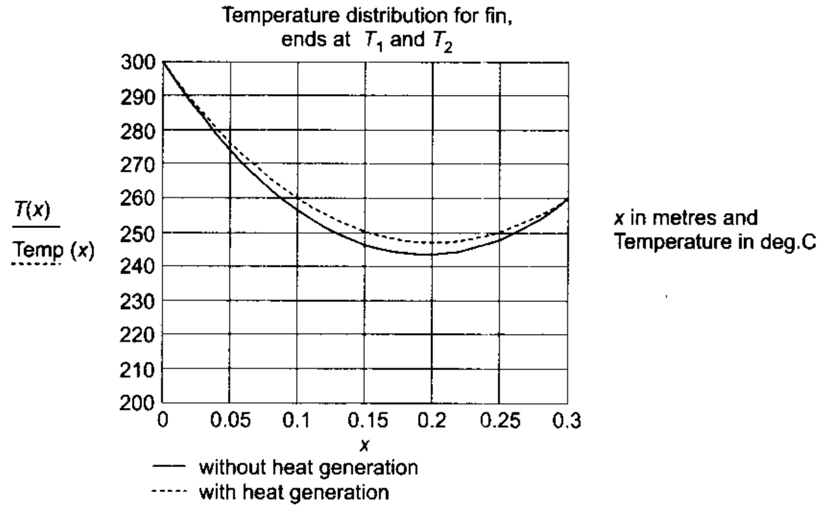
We have:

$$\theta(x) := 260 \cdot \cos h(m \cdot x) - 161.5218 \sin h(m \cdot x). \qquad ...(A)$$

Therefore, $T(x) := \theta(x) + T_a$ (*define T(x), tempreature at any point in the rod, **without heat generation.***)

$$\text{Temp}(x) := T_a + \frac{q_g}{k \cdot m^2} + \frac{\theta_1 \cdot (\sin h(m \cdot (L - x))) + \theta_2 \cdot \sin h(m \cdot x)}{\sin h(m \cdot L)}$$ (*define Temp(x) **with heat generation.***)

$x := 0, 0.01, ..., 0.3$ (*define a range variable x...starting value = 0, next value = 0.01 m, and last value = 0.3 m*)



Note: In the above graph, temperature distribution in the rod is drawn for both the cases i.e. with and without heat generation, for comparison. Temperature in the rod is everywhere higher with heat generation, as would be expected. With heat generation, minimum temperature occurs at 0.2 m from LHS, whereas without heat generation, minimum temperature occurs at 0.194 m from LHS. It can also be seen that left end is at 300°C and the right end at 260°C, as specified.

6.3 Fins of Non-uniform Cross Section

So far, we considered fins of uniform cross section. But, very often in practice, we find that fins of nonuniform cross section are also used. See Fig. 6.8, (b), (c), and (e). For example, annular fins are provided over a circular tube, as shown in Fig. 6.8(c), to enhance heat transfer. Here, the fin thickness may be constant, but its area for heat transfer along its radius, i.e. $(2\pi r t)$, varies with the radius.

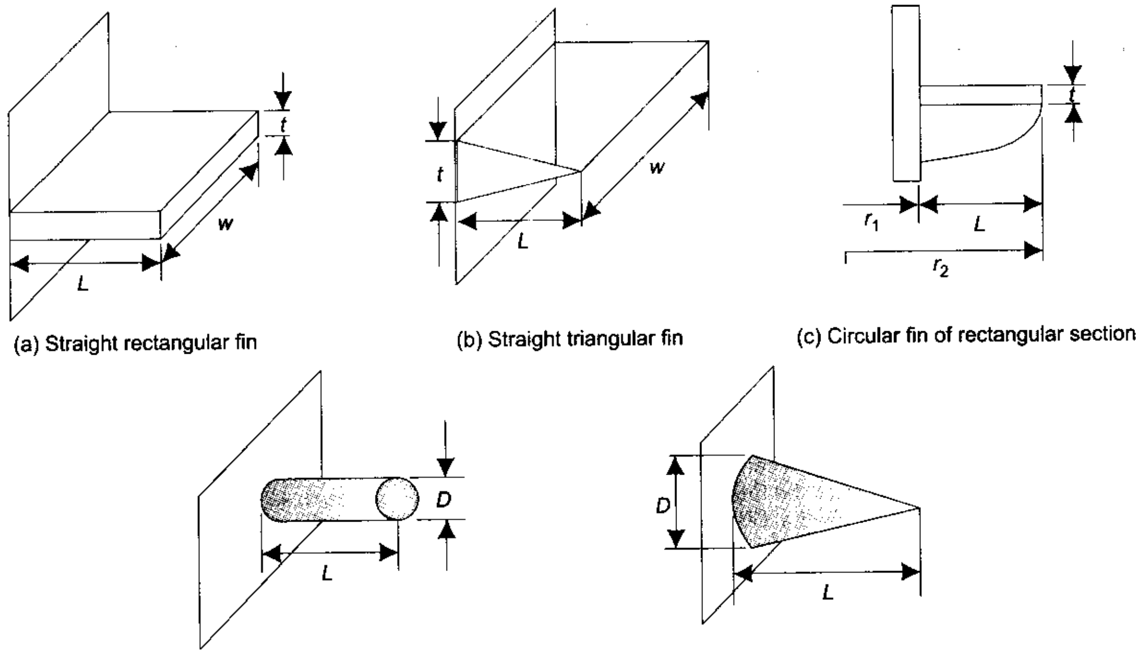
In such cases, the general differential equation governing the temperature distribution is derived by making an energy balance across an elemental volume, just as we did in the case of fin of uniform cross section.

Consider a fin of nonuniform cross section as shown in Fig. 6.9.

Consider an elemental section of thickness dx at a distance x from the base as shown. Let us write an energy balance for this element:

Energy going *into* the element by conduction = (Energy *leaving* the element by conduction + Energy *leaving* the surface of the element by convection)

$$\text{i.e. } Q_x = Q_{x+dx} + Q_{conv} \qquad ...(a)$$



(a) Straight rectangular fin (b) Straight triangular fin (c) Circular fin of rectangular section

(d) Pin fin, circular section (e) Pin fin, conical section

FIGURE 6.8 Typical Fins: (a) and (d) of uniform crosssection, and (b), (c) and (e): of non-uniform crosssection

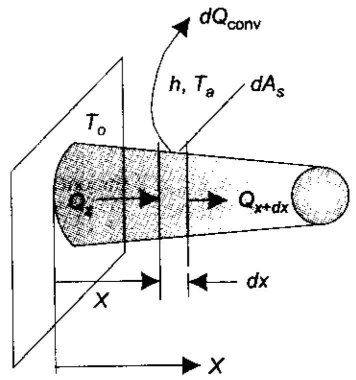


FIGURE 6.9 Fin of non-uniform cross-section

where,

Q_x = heat conducted into the element at x

Q_{x+dx} = heat conducted out of the element at $x + dx$, and

Q_{conv} = heat convected from the surface of the element to ambient.

We have, from Fourier's law:

$$Q_x = -k \cdot A_c \cdot \frac{dT}{dx}$$

Note that here, A_c, the cross-sectional area varies with x.

And,

$$Q_{x+dx} = Q_x + \frac{dQ_x}{dx} \cdot dx$$

i.e. $$Q_{x+dx} = -k \cdot A_c \cdot \frac{dT}{dx} - k \cdot \frac{d}{dx}\left(A_c \cdot \frac{dT}{dx}\right) \cdot dx$$

Convection heat transfer rate from the elemental volume is given by:

$$Q_{conv} = h \cdot dA_s \cdot (T - T_a)$$

where, dA_s is the surface area of the elemental volume.

Substituting the terms in Eq. a,

$$\frac{d}{dx} \cdot \left(A_c \cdot \frac{dT}{dx}\right) - \frac{h}{k} \cdot \frac{dA_s}{dx} \cdot (T - T_a) = 0$$

i.e. $$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c} \cdot \frac{dA_c}{dx}\right) \cdot \frac{dT}{dx} - \left(\frac{1}{A_c} \cdot \frac{h}{k} \cdot \frac{dA_s}{dx}\right) \cdot (T - T_a) = 0. \quad ...(6.15)$$

Equation 6.15 is the controlling differential equation for one-dimensional conduction in a fin of non-uniform cross section. Remember again that both A_c and A_s vary with x.

Solution of Eq. 6.15 gives the temperature distribution in the fin, and then, by applying Fourier's law, we can get the heat flux at any point.

However, solution of Eq. 6.15 is rather complicated and involves Bessel Functions. So, in practice, heat transfer in fins of non uniform cross sections are calculated by resorting to 'Fin efficiency' graphs, as explained in the next section.

6.4 Performance of Fins

Recollect that purpose of attaching fins over a surface is to increase the heat transfer rate. How well this purpose is achieved is characterised by two performance parameters:

(i) Fin efficiency, η_f and

(ii) Fin effectiveness, ε_f.

6.4.1 Fin Efficiency

Fin transfers heat to the surroundings from its surface, by convection. For convection heat transfer, the driving force is the temperature difference between the surface and the surrounding. However, temperature drops along the length of the fin because of the finite thermal conductivity of the fin material; so, heat transfer becomes less effective towards the end of the fin. Obviously, in the ideal case of the entire fin being at the same temperature as that of the base wall, the heat transferred from the fin will be maximum. So, fin efficiency is defined as the amount of heat actually transferred by a given fin to the ideal amount of heat that would be transferred if the entire fin were at its base temperature, i.e.

$$\eta_f = \frac{Q_{fin}}{Q_{max}} \quad ...(6.16)$$

where,

Q_{fin} = actual amount of heat transferred from the fin, and

Q_{max} = maximum (or ideal) amount of heat that would be transferred from the fin, if the entire fin surface were at the temperature of the base.

(a) For an infinitely long fin:

For an infinitely long fin, actual heat transferred is given by Eq. 6.5:

i.e. $$Q_{fin} = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_o = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot (T_o - T_a) \quad ...(6.5)$$

To calculate Q_{max}, if the entire fin surface were to be at a temperature of T_o, the convective heat transfer from the surface would be:

$$Q_{max} = h \cdot P \cdot L \cdot (T_o - T_a) \quad ...(A)$$

where, P is the perimeter of the fin section and (P.L) is the surface area of the fin.

Dividing Eq. 6.5 by Eq. A:

$$\eta_f = \frac{\sqrt{h \cdot P \cdot k \cdot A_c} \cdot (T_o - T_a)}{h \cdot P \cdot L \cdot (T_o - T_a)}$$

i.e. $$\eta_f = \frac{1}{\sqrt{\frac{h \cdot P}{k \cdot A_c}} \cdot L}$$

i.e. $$\eta_f = \frac{1}{m \cdot L}$$ ((6.17)...*fin efficiency for very long fin.*)

(b) For a fin with insulated end:

For the case of a fin with an insulated end, we get actual heat transferred Q_{fin} from Eq. 6.7:

i.e. $$Q_{fin} = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot (T_o - T_a) \cdot \tan h\,(m \cdot L) \quad ...(6.7)$$

and, fin efficiency is given by:

$$\eta_f = \frac{\sqrt{h \cdot P \cdot k \cdot A_c} \cdot (T_o - T_a) \cdot \tan h(m \cdot L)}{h \cdot P \cdot L \cdot (T_o - T_a)}$$

i.e.

$$\eta_f = \frac{\tan h(m \cdot L)}{\sqrt{\frac{h \cdot P}{k \cdot A_c}} \cdot L}$$

i.e.

$$\eta_f = \frac{\tan h(m \cdot L)}{m \cdot L}$$ ((6.18)...*fin efficiency for a fin with insulated end*)

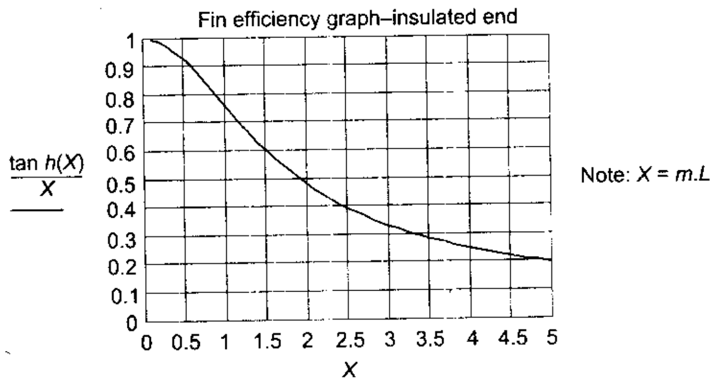
Note: For the more realistic case of a fin losing heat from its end, as stated earlier, to calculate heat transfer, Eq.6.9 itself may be used, but , with a corrected length L_c in place of L.

It is instructive to represent Eq. 6.18 in graphical form:

Let $X = m \cdot L$

$X := 0.1, 0.2.. 5$ (*let* $(m \cdot L)$ *vary from* $(m \cdot L) = 0.1$ *to* 5 *with an increment of* 0.1)

The graph looks as follows:



It may be noted from the graph that:

(i) Fin efficiency is maximum for the trivial case of $L = 0$, i.e. when there is no fin. So, fin efficiency is not maximised w.r.t. the fin length, but generally, w.r.t. volume or weight of material, which also has cost implications.

(ii) Nearer to the base of the fin, fin efficiency is high and it goes on decreasing as we move towards the end of the fin; this is because, the surface temperature of the fin falls as we move away from the base towards the end. Again , it is clear that there is not much gain if $(m \cdot L)$ is increased beyond a value of about 3.

Table 6.4 gives the values of fin efficiency for a few fin shapes:

Note: In Table 6.4:

I_0 = modified zero order Bessel function of first kind

K_0 = modified zero order Bessel function of second kind

I_1 = modified first order Bessel function of first kind

K_1 = modified first order Bessel function of second kind.

Fin efficiency graphs:

As can be seen from Table 6.4, expressions for fin efficiency of fins of non-uniform cross sections are rather complicated and involve Bessel functions. In practice, to find out the heat transfer from such fins, we use 'Fin efficiency charts'. Once the fin efficiency is obtained from these graphs, actual heat transferred from the fin is calculated using the definition of fin efficiency, i.e.

$$\eta_f = \frac{Q_{fin}}{Q_{max}}$$ ...(6.16)

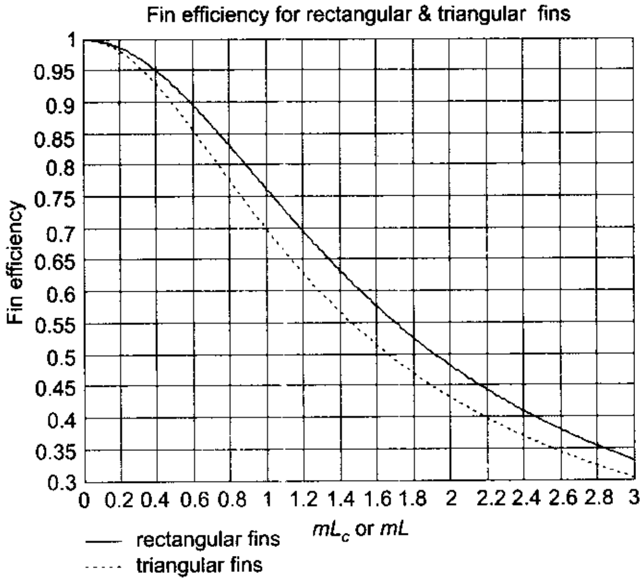
where,

$$Q_{max} = h \cdot P \cdot L \cdot (T_o - T_a)$$

TABLE 6.4 Fin efficiency (η_f) for a few fin shapes

A_c = area of cross section, A_f = total fin surface area, L_c = corrected length, P = perimeter of fin section, h = heat transfer coefficient, $m = \sqrt{\{h \cdot P/(k \cdot A_c)\}}$

Sl.No.	Description	Parameters	Fin efficiency (η_f)
1	Straight fin of rectangular section. See Fig. 6.8(a)	$A_f = 2 \cdot w \cdot L_c$ $L_c = L + \frac{t}{2}$ $m = \sqrt{\frac{2 \cdot h}{k \cdot t}}$ ...thin fins, $w >$	$\eta_f = \frac{\tan h(m \cdot L_c)}{m \cdot L_c}$
2	Straight fin of triangular section. See Fig. 6.8(b)	$A_f = 2 \cdot w \cdot \left[L^2 + \left(\frac{t}{2}\right)^2\right]^{\frac{1}{2}}$ $m = \sqrt{\frac{2 \cdot h}{k \cdot t}}$	$\eta_f = \frac{1}{m \cdot L} \cdot \frac{I_1(2 \cdot m \cdot L)}{I_o(2 \cdot m \cdot L)}$
3	Circular fin of rectangular section See Fig. 6.8(c)	$A_f = 2 \cdot \pi \cdot \left(r_{2c}^2 - r_1^2\right)$ $r_{2c} = r_2 + \frac{t}{2}$ $m = \sqrt{\frac{2 \cdot h}{k \cdot t}}$	$\eta_f = C_2 \cdot \left[\frac{(K_1(m \cdot r_1) \cdot I_1(m \cdot r_{2c}) - I_1(m \cdot r_1) \cdot K_1(m \cdot r_{2c}))}{(I_o(m \cdot r_1) \cdot K_1(m \cdot r_{2c}) + K_o(m \cdot r_1) \cdot I_1(m \cdot r_{2c}))}\right]$ $C_2 = \frac{\left(\frac{2 \cdot r_1}{m}\right)}{\left(r_{2c}^2 - r_1^2\right)}$
4	Pin fin, circular section See Fig. 6.8(d)	$A_f = \pi \cdot D \cdot L_c$ $L_c = L + \frac{D}{4}$ $m = \sqrt{\frac{4 \cdot h}{k \cdot D}}$	$\eta_f = \frac{\tan h(m \cdot L_c)}{m \cdot L_c}$
5	Pin fin, conical section See Fig. 6.8(e)	$A_f = \frac{\pi \cdot D}{2} \cdot \left[L^2 + \left(\frac{D}{2}\right)^2\right]^{\frac{1}{2}}$ $m = \sqrt{\frac{4 \cdot h}{k \cdot D}}$	$\eta_f = \frac{2}{m \cdot L} \cdot \frac{I_2(2 \cdot m \cdot L)}{I_1(2 \cdot m \cdot L)}$



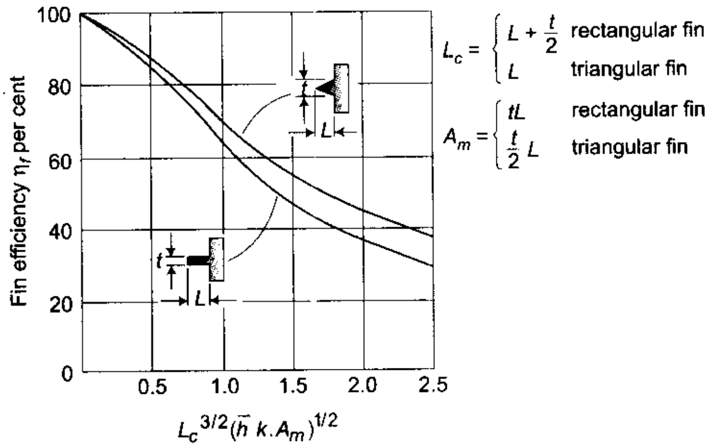


FIGURE 6.10 Efficiency of straight, rectangular (or, cylindrical, pin fins) and triangular fins (Ref. Fig.6.8, a & b)

Q_{max} is easily calculated from the given data.

Fig. (6.10) gives fin efficiency values for fins of rectangular and triangular sections.

It may be noted that graph for rectangular fins is also valid for cylindrical, pin fins since equation for fin efficiency is the same (see Table 6.4); of course, m and L_c will be different for pin fins.

For straight, rectangular fins and cylindrical, pin fins:

$$\eta_f = \frac{\tan h(m \cdot L_c)}{m \cdot L_c}$$

Let $X = m \cdot L_c$

i.e. $$\eta_f = \frac{\tan h(X)}{X}$$

For straight, triangular fins:

$$\eta_f = \frac{1}{m \cdot L} \cdot \frac{I_1(2 \cdot m \cdot L)}{I_o(2 \cdot m \cdot L)}$$

Let $Z = m \cdot L$

i.e. $$\eta_f = \frac{1}{Z} \cdot \frac{I_1(2 \cdot Z)}{I_o(2 \cdot Z)}$$

TABLE 6.5 Efficiency vs. mL_c for straight, rectangular fins, or cylindrical pin fins $X = m \cdot L_c$

$$\text{Effcy}(X) = \frac{\tan h(X)}{X} \quad X = 0.1, 0.2, \ldots, 3$$

X	$\frac{\tan h(X)}{X}$
0.1	0.997
0.2	0.987
0.3	0.971
0.4	0.95
0.5	0.924
0.6	0.895
0.7	0.863
0.8	0.83
0.9	0.796
1	0.762
1.1	0.728
1.2	0.695
1.3	0.663
1.4	0.632
1.5	0.603
1.6	0.576
1.7	0.55
1.8	0.526
1.9	0.503
2	0.482
2.1	0.462
2.2	0.444
2.3	0.426
2.4	0.41
2.5	0.395
2.6	0.38
2.7	0.367
2.8	0.355
2.9	0.343
3	0.332

TABLE 6.6 Efficiency vs. mL for straight, triangular fins $Z = m \cdot L$

$$\text{Effcy}(Z) = \frac{1}{Z} \cdot \frac{I_1(2 \cdot Z)}{I_0(2 \cdot Z)} \quad Z = 0.1, 0.2, \ldots, 3$$

Z	$\frac{1}{Z} \cdot \frac{I_1(2 \cdot Z)}{I_0(2 \cdot Z)}$
0.1	0.995
0.2	0.981
0.3	0.958
0.4	0.928
0.5	0.893
0.6	0.855
0.7	0.815
0.8	0.775
0.9	0.736
1	0.698
1.1	0.662
1.2	0.628
1.3	0.597
1.4	0.567
1.5	0.54
1.6	0.515
1.7	0.492
1.8	0.47
1.9	0.45
2	0.432
2.1	0.415
2.2	0.399
2.3	0.384
2.4	0.37
2.5	0.357
2.6	0.345
2.7	0.334
2.8	0.323
2.9	0.313
3	0.304

Note: On x-axis use: $(m \cdot L_c)$ for rectangular fins and, $(m \cdot L)$ for triangular fins

Since these two types of fins are used very much in practice, efficiency values are also given in tabular form above:

Fig. 6.11 gives fin efficiency values for circumferential fins of rectangular profile.

Note that X-axis in Fig. 6.10 is:

$$L_c^{\frac{3}{2}} \cdot \sqrt{\frac{h}{k \cdot A_m}}$$

where, A_m is the profile area of the fin.

($A_m = L.t$ for rectangular section and $(L.t/2)$ for a triangular section).

Rationale of using this complicated expression in the X-axis is as follows:

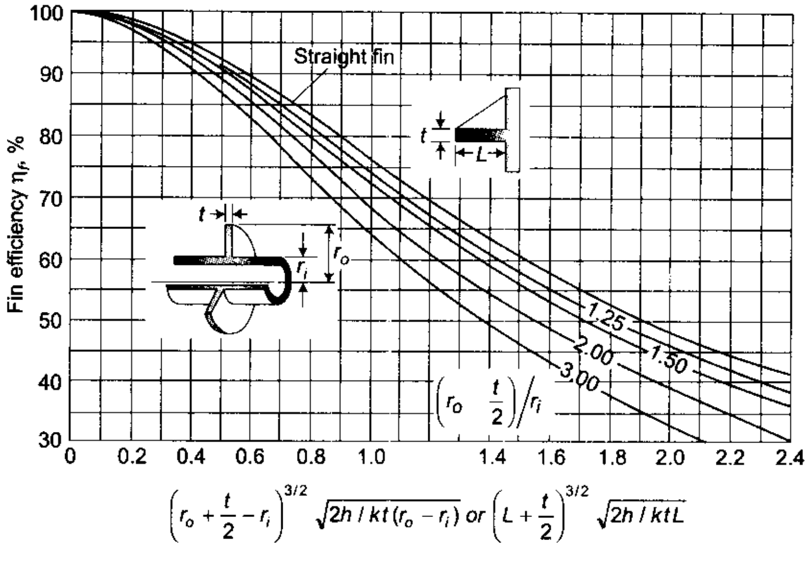


FIGURE 6.11 Efficiency of circumferential rectangular fins

Considering a fin of rectangular cross section, insulated at its end, we can write:

$$\eta_f = \frac{\tan h(m \cdot L)}{m \cdot L}$$

Now, $$m \cdot L = \sqrt{\frac{h \cdot P}{k \cdot A_c}} \cdot L = \sqrt{\frac{h \cdot (2 \cdot w + 2 \cdot t)}{k \cdot w \cdot t}} \cdot L$$

For a very wide fin: i.e. $w >> t$, we can write:

$$m \cdot L = \sqrt{\frac{2 \cdot h \cdot w}{k \cdot w \cdot t}} \cdot L = \sqrt{\frac{2 \cdot h}{k \cdot t}} \cdot L = \sqrt{\frac{2 \cdot h}{k \cdot t \cdot L}} \cdot L^{\frac{3}{2}}$$

i.e. $$m \cdot L = \sqrt{\frac{2 \cdot h}{k \cdot A_m}} \cdot L^{\frac{3}{2}} \qquad ...(6.19)$$

where, $A_m = (L.t)$, is the profile area for the rectangular section. So, on the X-axis, instead of $(m.L)$, what is plotted is:

$$L_c^{\frac{3}{2}} \cdot \sqrt{\frac{h}{k \cdot A_m}}$$

where, L_c is the corrected length, to take into account convection from the end.

6.4.2 Fin Effectiveness (ε_f)

Consider a fin of uniform cross-sectional area A_c, fixed to a base surface. Purpose of the fin is to enhance the heat transfer. If the fin were not there, heat would have been transferred from the base area A_c, by convection. By attaching the fin, area for convection increases i.e. convective resistance ($= 1/(h.A)$) decreases; however, it is obvious that a conduction resistance due to the solid fin is now introduced and the total heat transfer would depend upon the net thermal resistance. As we go on increasing the length of fin, convection resistance will go on decreasing, but conduction resistance will go on increasing. This means that attaching a fin may not necessarily result in effectively increasing the heat transfer. Therefore, how effective the fin is in enhancing the heat transfer is characterised by a parameter called fin effectiveness.